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DYNAMIC STABILITY OF STRUCTURES: APPLICATION TO FRAMES, CYLINDR--ETC(U)
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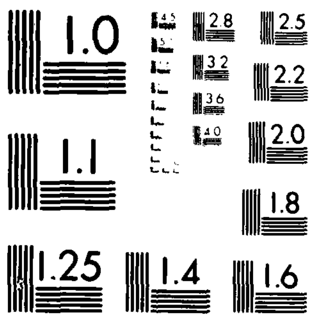

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DYNAMIC STABILITY OF STRUCTURES: APPLICATION TO FRAMES, CYLINDRICAL SHELLS AND OTHER SYSTEMS

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FEBRUARY 1982

TECHNICAL REPORT AFWAL-TR-81-3155
Final Report for Period May 1980 - April 1981

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFWAL-TR-81-3155	2. GOVT ACCESSION NO. AD-A114 735	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DYNAMIC STABILITY OF STRUCTURES: APPLICATION TO FRAMES, CYLINDRICAL SHELLS AND OTHER SYSTEMS		5. TYPE OF REPORT & PERIOD COVERED Final Report May 1980 - April 1981
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) George J. Simitses, Izhak Sheinman		8. CONTRACT OR GRANT NUMBER(s) F33615-79-C-3221
9. PERFORMING ORGANIZATION NAME AND ADDRESS Georgia Institute of Technology Atlanta, Georgia 30332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P.E. 62201F Proj. 2401, Task 240102 Work Unit 24010228
11. CONTROLLING OFFICE NAME AND ADDRESS Flight Dynamics Laboratory (AFWAL/FIBRA) Air Force Wright Aeronautical Laboratories (AFSC) Wright-Patterson AFB, Ohio 45433		12. REPORT DATE February 1982
		13. NUMBER OF PAGES 197
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Dynamic Stability
Frames
Arches
Cylindrical Shells
Imperfection Sensitivity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report deals primarily with extension of the energy-based concepts of dynamic stability, developed earlier for finite-degree-of-freedom systems, to continuous systems. Moreover, the related criteria for dynamic stability are demonstrated through several structural configurations, such as eccentrically loaded simple two-bar frames, geometrically imperfect, thin, cylindrical shells (of stiffened and unstiffened construction) and subjected to uniform axial compression and lateral pressure, and a pinnted, half-sine, shallow arch

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loaded transversely. All of these systems are subject to violent buckling under static application of the loads. Moreover, the developed concepts are extended, so as to apply to structural systems, which are either subject to smooth buckling or are not subject to buckling at all under static loading. This extension is clearly demonstrated through several simple examples. Through this extension it is shown that, in a general sense for systems loaded by sudden loads, there is no question of dynamic stability or instability, but only a question of dynamic response in a deflectional space with imposed limitation on the size of the allowable deflections. Finally, in the case of the imperfect cylindrical shell, in order to find critical dynamic conditions, it was necessary to develop a solution scheme that describes completely the behavior of the shell under static loading (including post-limit point behavior). The solution scheme and the related computer program are fully explained.

FOREWORD

This report was prepared by Professor George J. Simitses and Dr. Izhak Sheinman, both of the school of Engineering Science and Mechanics at the Georgia Institute of Technology, Atlanta, Georgia. Dr. Sheinman is a visiting scholar, on leave from the Department of Civil Engineering of the Technion-Israel Institute of Technology, Haifa, Israel. This work was performed under Contract No. F33615-79-C-3221 with the U.S.A.F. Aeronautical Systems Division (AFSC), Wright-Patterson Air Force Base, Ohio 45433. This final technical report was released by the authors in June 1981. The report covers work conducted under contract, from May 1980 through April 1981 (end of contract period).

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SECTION I

INTRODUCTION

This report, in essence, is a continuation of Ref. 1. In this reference, the concept of dynamic stability for suddenly loaded systems was discussed in detail, including criteria and estimates, on the basis of the energy approach. Moreover, they were demonstrated through simple mechanical models, with finite-degrees-of-freedom. These models are characteristic of imperfection sensitive systems, and under static conditions they are subject to either limit-point instability or unstable bifurcational instability (in both cases, violent buckling). The suddenly applied loads are of constant magnitude and, in general, of finite duration. The extreme cases of ideal impulse and constant load of infinite duration were discussed and presented separately, as well, for two reasons: (a) the concepts for these two cases are simpler to present and (b) historically these extreme cases were treated extensively by the initial investigators of dynamic stability of suddenly-loaded structures [2-13]. The methodologies employed by these investigators can be classified into three groups: (i) equations of motion approach (Budiansky-Roth [3]), (ii) the phase plane-total energy approach (Hsu [6]), and (iii) the potential energy approach (Hoff-Simitses [2,11]).

The first approach (a) has had wide acceptance, because it is well suited for computer-type solutions. The equations of motion are solved for various levels of the load parameter. For small levels, the solution is simply oscillatory; as the load level increases, the motion changes to distinctly large amplitude (from the initial undisturbed position) oscillations. The load level at which this change occurs is termed

critical dynamic load. A great advantage of this approach is that it can estimate the critical conditions accurately, and the loading can be, in general, time-dependent. Of course, the obvious disadvantage is that it is extremely difficult and it requires a large amount of computer time to solve the equations of motion for various levels of the applied load (constant of infinite duration). The difficulty and the required time further increase as the load becomes time-dependent.

The other two approaches can estimate conditions under which the motion will remain oscillatory about the near static equilibrium position (lower bound on critical suddenly applied load) or conditions under which the motion will definitely be of large amplitude (upper bound). Hsu termed the former a sufficiency condition for stability and the latter a sufficiency condition for instability. Simitses referred to these two bounds as critical loads and he termed the former minimum possible critical load (MPCL) and the latter minimum guaranteed critical load (MGCL). In many cases considered, the static critical load is coincident with the upper bound. Needless to say, that the emphasis of approaches (b) and (c) is to estimate the lower bound and use this as a basis for design (whenever applicable).

This report deals primarily with extension of the energy concepts discussed in Ref. 1, and application of the related criteria to a number of practical structural configurations, such as frames, imperfect cylindrical shells (of stiffened and unstiffened construction) and shallow arches. These systems also are subject to violent buckling under static application of the loads.

Moreover, in the last chapter of the present report, the developed concepts are applied to structural systems which are not subject to

violent buckling under quasistatic application of the loads. The ensuing discussion provides the necessary clarifications.

All structural configurations, when acted upon by quasi-static loads, respond in a manner described in one of the five figures, Figs. 1.1 - 1.5. These figures characterize equilibrium positions (structural response) as plots of a load parameter, P, versus some characteristic displacement, θ. The solid curves denote the response of systems which are free of imperfections and the dashed-line curves denote the response of the corresponding imperfect configuration.

Fig. 1.1 shows the response of such structural elements as columns, plates, and unbraced portal frames. The perfect configuration is subject to bifurcational buckling, while the imperfect configuration is characterized by stable equilibrium (unique), for elastic material behavior.

Fig. 1.2 typifies the response of some simple trusses and two-bar frames. The perfect configuration is subject to bifurcational buckling, but smooth (stable branch) in one direction of the response and violent (unstable branch) in the other. Correspondingly the response of the imperfect configuration is characterized by stable equilibrium (and unique) for increasing load in one direction, while in the other, the system is subject to limit point instability.

Fig. 1.3 typifies the response of troublesome structural configurations such as cylindrical shells (especially under uniform axial compression and of isotropic construction), pressure-loaded spherical shells and some simple two-bar frames. These systems are imperfection-sensitive systems and are subject to violent buckling under static loading.

A large class of structural elements is subject to limit point instability. In some cases, unstable bifurcation is present in addition

to the limit point. The response of such systems is shown on Fig. 1.4. Two structural elements that behave in this manner are the shallow spherical cap and the low arch. Both elements have been used extensively.

Finally, there is a very large class of structural elements, which are always in stable equilibrium for elastic behavior and for all levels of the applied loads. These systems are not subject to instability under static conditions. Typical members of this class are beams, and transversely loaded plates. For this class of structural elements, Fig. 1.5 shows a typical load-displacement curve.

The concept of dynamic stability as developed in Ref. 1 and as discussed in Refs. 3 and 6, was always with reference to systems which under static loading are subject to violent buckling. This implies that dynamic buckling has been discussed for systems with static behavior shown in Figs. 1.2 (to the left), 1.3 and 1.4.

In developing concepts and the related criteria and estimates for dynamic buckling (see Ref. 1), it was observed that, even for systems which are subject to violent (static) buckling, critical dynamic loads can be associated with limitations in deflectional response rather than escaping motion through a static unstable point. This is especially applicable to the design of structural members and configurations, which are deflection limited. From this point of view then, the concept of dynamic stability can be extended to all structural systems especially those of Figs. 1.1 (imperfect), 1.2 (imperfect and to the right), and 1.5. Note that from this point of view there is no question of dynamic instability, but strictly a question of dynamic response in a limited deflection space.

This extension of the concept will be amplified in Section 5. The examples and applications chosen clarify the extension. Moreover, the

problem of a suddenly loaded imperfect column is presented, primarily because it represents the only system, other than those which are subject to violent (static) buckling, which has received some attention in the open literature.

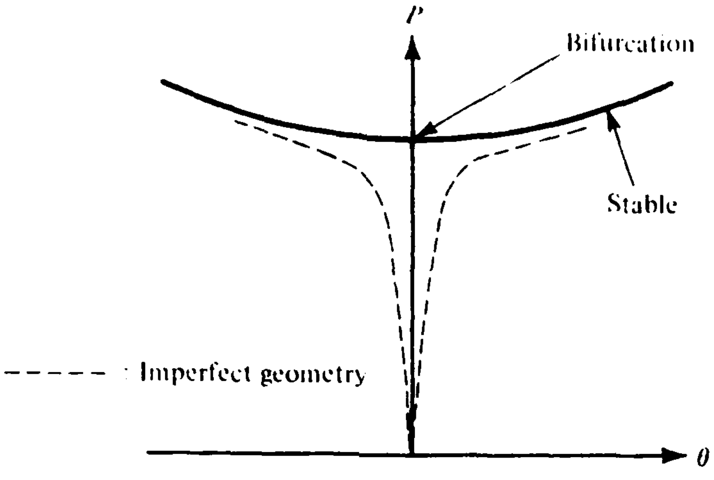

Fig. 1.1. Bifurcated Equilibrium Paths with Stable Branching.

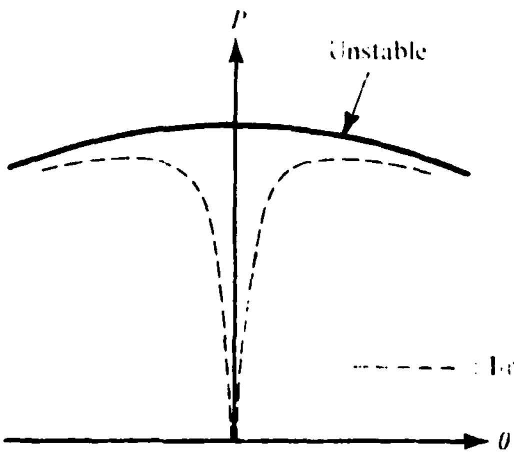

Fig. 1.3. Bifurcated Equilibrium Paths with Unstable Branching.

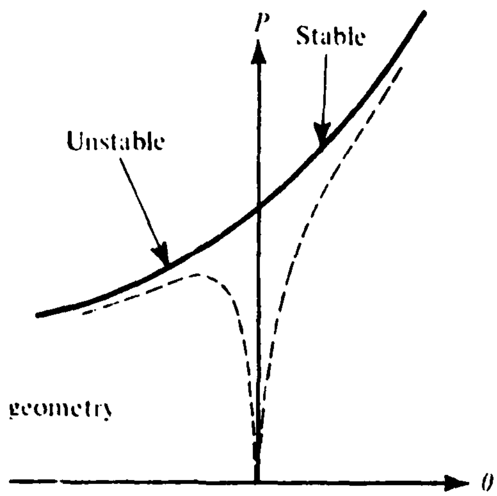

Fig. 1.2. Bifurcated Equilibrium Paths with Stable and Unstable Branches.

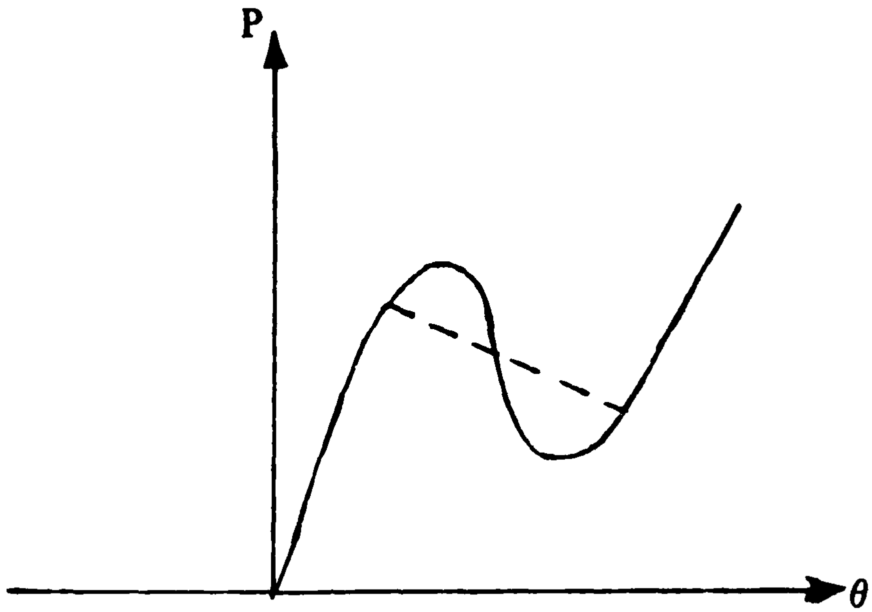

Fig. 1.4. Snap-through Buckling Paths
(Through Limit Point or Unstable Branching).

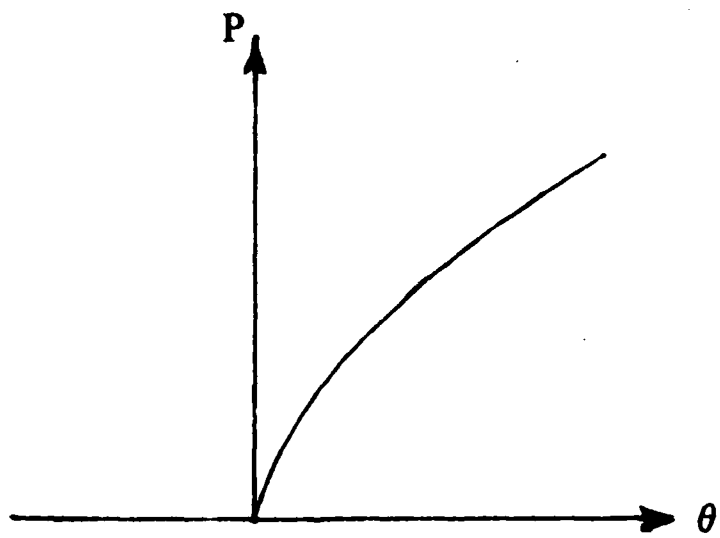

Fig. 1.5. Unique Stable Equilibrium Path.

SECTION II

SIMPLE TWO-BAR FRAMES UNDER SUDDENLY APPLIED LOADS

The Stability Criterion

Consider the simple two-bar frame shown of Fig. 2.1. The two bars are of the same structural geometry (length ℓ, corss-sectional area A, second moment of area I, and Young's modulus E). The vertical bar is supported by an immovable hinge, while the horizontal bar is supported by a hinge with three variations: (a) immovable (Model A), (b) movable in a vertical direction (Model B), and (c) movable in a horizontal direction (Model C). The external load, $P(t) = H(t)P$ [$H(t)$ is the heaviside function], is applied vertically with an eccentricity e, and it represents a constant force, P, suddenly applied, with infinite duration. The transverse and axial displacement components are $w_i(x,t)$ and $\xi(x,t)$, respectively.

By employing Hamilton's principle, one can obtain the equations of motion for the system.

$$\delta \int_{t_1}^{t_2} (T - U_T)\, dt = 0 \tag{2.1}$$

where the functionals U_T (total potential) and T (kinetic energy) are given by

$$U_T\left[\xi_i, w_i ; P\right] = \frac{1}{2} \sum_{i=1}^{2} \int_0^{\ell} \left[AE\left(\xi_{i,x} + \frac{1}{2} w_{i,x}^2\right)^2 + EI\, w_{i,xx}^2\right] dx$$

$$+ H(t)\left[P\, \xi_1(\ell) + Pe\, w_{2,x}(\ell)\right] \tag{2.2}$$

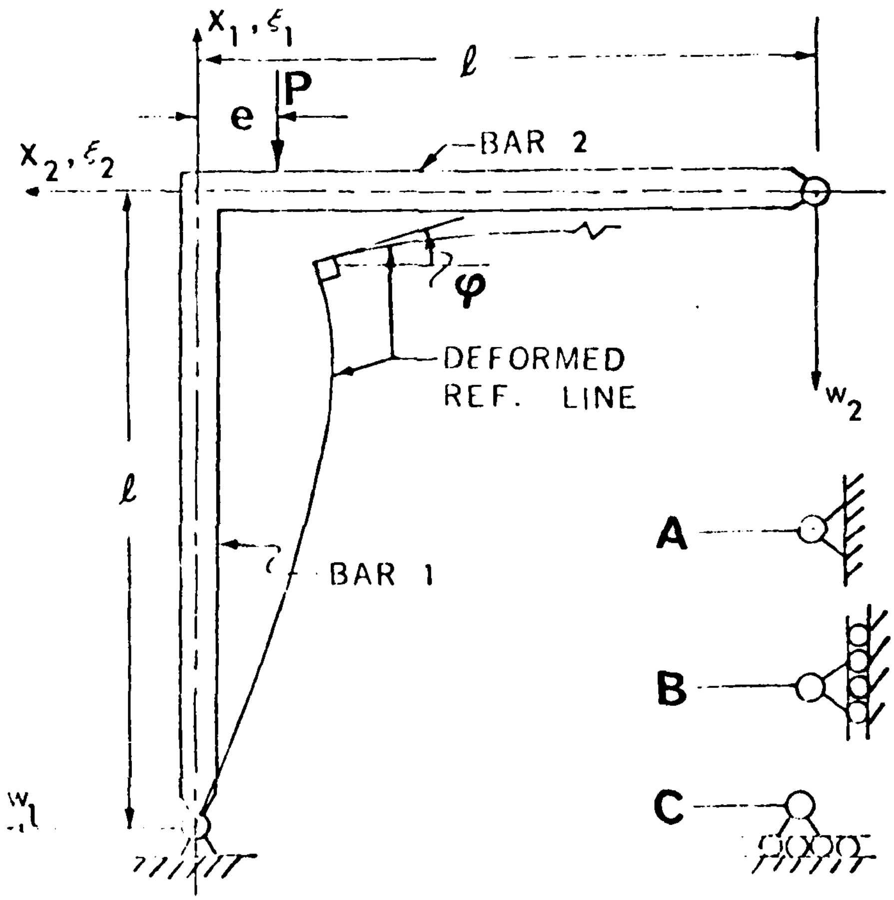

Fig. 2.1 Geometry and Sign Convention of a Two-Bar Frame

$$T\left[\xi_i, w_i ; P\right] = \frac{1}{2} \sum_{i=1}^{2} \int_0^{\ell} m \left(w_{i,t}^2 + \xi_{i,t}^2\right) dx \tag{2.3}$$

where the indices denote partial derivatives and m is the mass per unit length, the same for both bars. The equations of motion are:

$$EA \left(\xi_{i,x} + \frac{1}{2} w_{i,x}^2\right)_{,x} - m\, \xi_{i,tt} = 0$$

$$EI\, w_{i,xxxx} - EA \left[\left(\xi_{i,x} + \frac{1}{2} w_{i,x}^2\right) w_{i,x}\right]_{,x} + m w_{i,tt} = 0, \quad i = 1,2 \tag{2.4}$$

The kinematic continuity conditions at the joint are

$$w_1(\ell,t) = \xi_2(\ell,t); \; w_2(\ell,t) = -\xi_1(\ell,t);$$

$$w_{1,x}(\ell,t) = w_{2,x}(\ell,t) \tag{2.5}$$

The natural boundary conditions at the joint are obtained from the variational problem and by employing Eqs. (2.5), which are independent of this formulation. These natural boundary (force) conditions are:

$$AE\left[\xi_{1,x}(\ell,t) + \frac{1}{2} w_{1,x}^2(\ell,t)\right] + EI\, w_{2,xxx}(\ell,t) - AE\left[\xi_{2,x}(\ell,t) + \frac{1}{2} w_{2,x}^2(\ell,t)\right] w_{2,x}(\ell,t) + PH(t) = 0$$

$$AE\left[\xi_{1,x}(\ell,t) + \frac{1}{2} w_{1,x}^2(\ell,t)\right] w_{1,x}(\ell,t) - EI\, w_{1,xxx}(\ell,t) + AE\left[\xi_{2,x}(\ell,t) + \frac{1}{2} w_{2,x}^2(\ell,t)\right] = 0 \tag{2.6}$$

$$EI\left[w_{1,xx}(\ell,t) + w_{2,xx}(\ell,t)\right] + H(t)\, Pe = 0$$

Finally, the support conditions for the three models (A,B and C) are listed below, as common to both models and those, which are different

<u>Common</u>

$$\xi_1(0,t) = W_1(0,t) = W_{1,xx}(0,t) = W_{2,xx}(0,t) = 0 \tag{2.7}$$

<u>Model A</u>

$$\xi_2(0,t) = 0, \qquad W_2(0,t) = 0 \tag{2.8}$$

Model B

$$\xi_2(0,t) = 0, \quad EI\, w_{2,xxx}(0,t) - AE\left[\xi_{2,x}(0,t) + \frac{1}{2} w_{2,x}^2(0,t)\right] = 0 \tag{2.9}$$

Model C

$$w_2(0,t) = \xi_{2,x}(0,t) + \frac{1}{2} w_{2,x}^2(0,t) = 0 \tag{2.10}$$

The initial conditions must reflect the fact that, the system is at rest initially ($t = 0$).

$$\xi_i(x,0) = w_i(x,0) = 0 \tag{2.11}$$

$$\xi_{i,t}(x,0) = w_{i,t}(x,0) = 0 \qquad i = 1,2$$

The solution of the equations of motion, Eqs. (2.4), a system of coupled nonlinear partial differential equations, subject to the auxiliary, Eqs. (2.5) - (2.7) and (2.8), or (2.9), or (2.10), and initial conditions, Eqs. (2.11), is, at best, extremely difficult. Furthermore, even if the solution is possible, for various magnitudes of the applied force, P, of the eccentricity, e, and slenderness ratio, λ, a criterion for stability is still missing. One possibility, here, is to employ the criterion of Budiansky and Roth [3], which, in this particular case, requires a wise choice for a characteristic displacement response. Another possibility, of course, is to employ this criterion in conjunction with an approximate solution obtained on the basis of either direct variational methods or methods of weighted residuals [14]. In this latter approach, there is an uncertainty with both the direction and the estimation of the error involved in the approximation.

In the light of the above difficulties, a simpler approach, giving accurate results for design purposes, is needed. The analysis, described and employed herein, provides an extension of the energy approach as developed in Ref. 1. This extension is described in the ensuing paragraphs and, in so doing, criteria and estimates are clearly established.

First, by virtue of the initial conditions, Eqs. (2.11), both the kinetic energy, T, and the total potential, U_T, are zero at t = 0. From the law of conservation of energy, for this undamped conservative system, initially stress free, the total energy (Hamiltonian) is constant for t > 0.

$$U_T\left[\xi_i,\ w_i\ ;\ P\right] + T\left[\xi_{i,x}\ w_i\right] = C \tag{2.12}$$

where U_T and T are given by Eqs. (2.2) and (2.3), respectively, and C is a known constant (this constant can be made zero by properly defining U_T).

Since T is positive definite, Eq. (2.3), for all kinematically admissible trajectories, then motion is possible only when U_T is non-positive, or

$$U_T\left[\xi_i,\ w_i\ ;\ P\right] \le 0 \tag{2.13}$$

Next, let us assume that, at some level of the applied load $P = \bar{P}$, the ensuing motion is bounded, for all kinematically admissible trajectories. Then, for each possible trajectory, regardless of whether it corresponds to a true trajectory of motion or not, there is a set of displacement functions of position $\left[\overset{*}{\xi}_i(x),\ \overset{*}{w}_i(x)\right]$, i = 1,2 (because of

bounded motion) such that

$$U_T \left[\xi_i, w_i ; \bar{P}\right] = 0 \tag{2.14}$$

The collection of all such sets, corresponding to all conceivable, kinematically admissible trajectories, forms a boundary, which is characterized by $U_T = 0$ and it separates the region of $U_T < 0$ from the region of $U_T > 0$. This boundary is dependent upon the level of the applied load, $\bar{P}$. Because of Eq. (2.13), it has been established that motion can only take place in the region characterized by $U_T < 0$. At this point, note that, regardless of the trajectory, the velocities ($\xi_{i,t}$, $w_{i,t}$) are zero and a change in the motion takes place, whenever the system reaches the boundary described above. Furthermore, the region characterized by $U_T < 0$ must contain at least one relative minimum point for U_T, regardless of trajectory and time. It may contain more than one relative minimum and some other stationary points. These stationary points are characterized by sets of displacement magnitudes and shapes, which, by definition, correspond to static equilibrium positions for the particular value of the applied load, P.

By having set, thusly, the stage, we can now define "unbuckled" motion (see also [1]). If the bounded region ($U_T < 0$) contains only one stationary point (a minimum) then the motion for this $\bar{P}$ is called "unbuckled". The physical interpretation of "unbuckled" motion corresponds to the system performing nonlinear oscillations about the corresponding near stable static equilibrium position. This is exactly the case for small levels of the suddenly applied load. The only way the motion can become "buckled" (unbounded in the sense described above) is if the

boundary characterized by the set of $\left[\overset{*}{\xi}_i(x),\ \overset{*}{w}_i(x)\right]$ contains an unstable static equilibrium position for the corresponding value of the applied load. This value of the load represents an upper bound of all loads for which the motion remains "unbuckled", and it is called, herein, critical dynamic load. Incidentally, this level of the load corresponds to Hsu's sufficiency condition for stability and Simitses' minimum possible critical load (MPCL). Note that when damping is present, for loads smaller than this critical dynamic load, the system is asymptotically stable, because it will eventually come to rest at the near static equilibrium position.

For the frame problems, under consideration, one needs only solve the corresponding static problem and compute the value of the total potential U_T, at every static unstable equilibrium point. This static solution is outlined below and it is given in detail in [15, 16]. By dropping the inertia terms, the equations of motion, Eqs. (2.4), become static equilibrium equations. The general solutions, obtained from these equations, for the four displacement components, w_i, ξ_i ($i = 1, 2$), are computed in terms of simple spatial functions and twelve constants. Use of the three kinematic continuity conditions Eqs. (2.5), the three natural boundary conditions at the joint, Eqs. (2.6) and the six support conditions, Eqs. (2.7) - (2.11), yields a system of twelve equations in the twelve constants. The load parameter, the eccentricity and the slenderness ratio appear also in these equations. Some of these constants are zero and the remaining ones, except for two, appear in a linear sense. Elimination of these particular constants finally yields a system of, at most, two non-linear equations that relate two constants to the

given parameters. A methodology is outlined in [15] for solving these two nonlinear equations.

It should be mentioned here that the two constants appearing in the nonlinear equations are measures of the axial force in the two bars (k_i) [the complete solutions are given in the next section). The given parameters are listed below in nondimensionalized form

$$\beta^2 = \frac{P\ell^2}{EI} ; \; \bar{e} = \frac{e}{\ell} ; \; \lambda = \frac{\ell}{\rho} \tag{2.15}$$

where $\rho^2 = I/A$.

Thus, the total potential, U_T, for the static problem may be expressed solely in terms of k_i, b^2, $\bar{e}$ and λ, i.e.

$$U_T = U_T \; (k_i, \beta^2, e, \lambda) \tag{2.16}$$

For any given geometry (e, λ), static equilibrium positions are shown as plots of load parameter, β^2, versus joint rotations (characteristic displacement - see Fig. 2,2). At every point of this curve, the value of the total potential is computed. The value of β^2 at which U_T changes from negative to positive corresponds to the dynamic critical load, $\beta^2_{cr_D}$. Another possible procedure for finding $\beta^2_{cr_D}$ is the simultaneous solution of the two nonlinear equilibrium equations

$$\frac{\partial U_T}{\partial k_i} = 0, \quad k_i = 1,2 \tag{2.17}$$

and

$$U_T = 0 \tag{2.18}$$

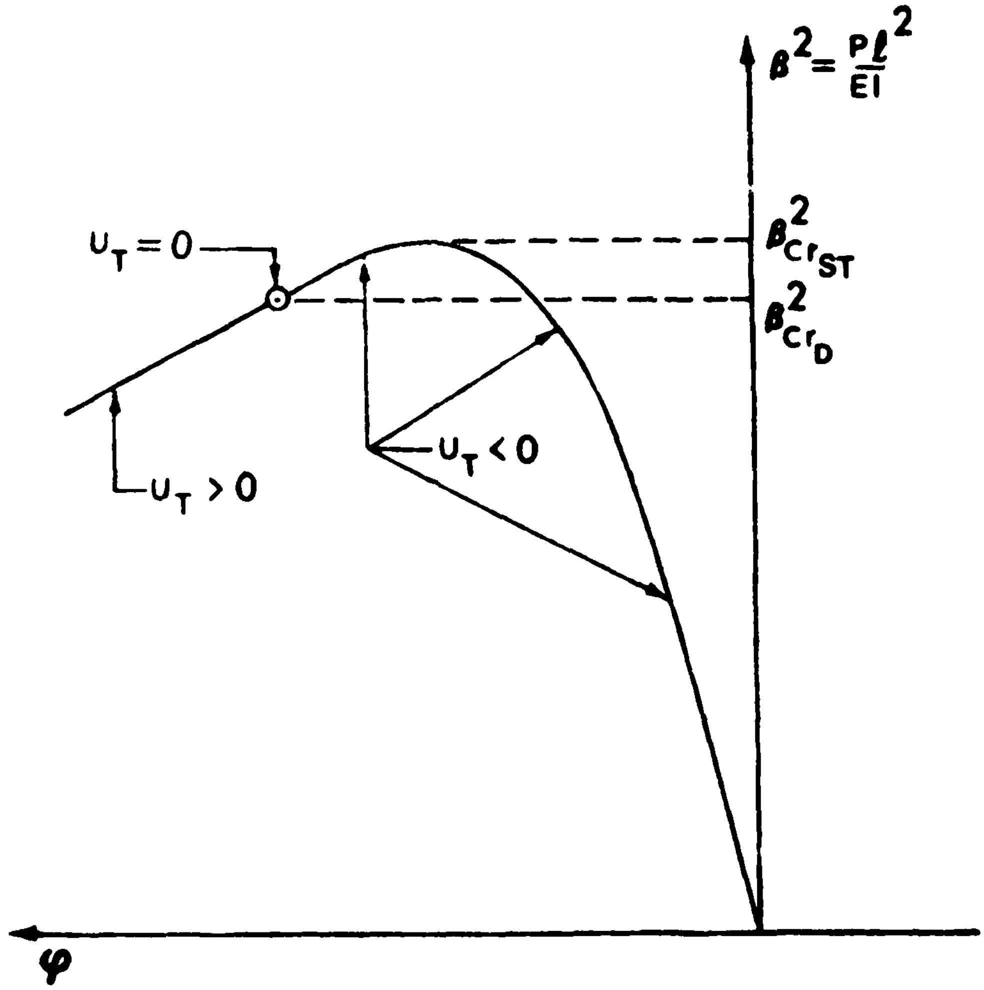

Fig. 2.2. Typical Load versus Characteristic Displacement (Joint Rotation) Curve.

subject to the condition that the static equilibrium position, at which $U_T = 0$, is unstable, or

$$\frac{\partial^2 U_T}{\partial k_i^2} \not> 0 \qquad i = 1, 2$$

$$\left(\frac{\partial^2 U_T}{\partial k_1^2}\right)\left(\frac{\partial^2 U_T}{\partial k_2^2}\right) \not> \left(\frac{\partial^2 U_T}{\partial k_1 \partial k_2}\right)^2 \tag{2.19}$$

Note that Eqs. (2.19) imply that, the conditions for static stability are violated.

This alternate procedure needs special attention, because it can easily yield physically unacceptable solutions. These unacceptable solutions arise from the nonlinearity of the problem and do not belong on the load-displacement curve shown on Fig. 2.2 (physically unacceptable).

The numerical results, presented herein, are obtained from the first solution, although the alternate solution was employed for spot checking. Data are generated on the Georgia Tech high speed digital computer CDC-Cyber-70, Model 74-28.

Summary of Static Solution Expressions

In addition to the nondimensionalized parameters given by Eqs. (2.15), the following nondimensionalized parameters are also introduced.

$$X = x/\ell; \qquad w_i = w_i/\ell; \qquad \Xi_i = \xi_i/\ell$$

$$k_i^2 = \frac{S_i \ell^2}{EI} \qquad i = 1, 2 \tag{2.20}$$

where S_i is the magnitude of the compressive force in the ith bar.

With these parameters the solution to the corresponding static problem and the expression for U_T are given below for each model separately.

<u>Model A</u>

$$\Xi_1(X) = -\frac{k_1^2}{\lambda^2}X - B_1(X)$$

$$W_1(X) = A_{11}\sin k_1X + A_{13}X \tag{2.21}$$

$$\Xi_2(X) = -\frac{k_2^2}{\lambda^2}X - B_2(X)$$

$$W_2(X) = A_{21}\sin k_2x + A_{23}X$$

where

$$B_1(X) = \frac{1}{2}\left[A_{13}^2X + 2A_{11}A_{13}\sin k_1X + \frac{k_1^2A_{11}^2}{2}\left(X + \frac{\sin 2k_1X}{2k_1}\right)\right]$$

$$B_2(X) = \frac{1}{2}\left[A_{23}^2X + 2A_{21}A_{23}\sin k_2X + \frac{k_2^2A_{21}^2}{2}\left(X + \frac{\sin 2k_2X}{2k_2}\right)\right] \tag{2.22}$$

$$A_{11} = \frac{\beta^2\bar{e}k_2\cos k_2 + \left(k_1^4 + k_2^4 - k_1^2\beta^2\right)\sin k_2/k_1^2}{k_1k_2(k_1\sin k_1\cos k_2 + k_2\cos k_1\sin k_2)} \;;$$

$$A_{21} = \frac{\beta^2\bar{e}k_1\cos k_2 + \left(k_1^4 + k_2^4 - k_1^2\beta^2\right)\sin k_2/k_2^2}{k_1k_2(k_1\sin k_1\cos k_2 + k_2\cos k_1\sin k_2)}$$

$$A_{13} = -k_2^2/k_1^2 \quad ; \quad A_{23} = \left(k_1^2 - \beta^2\right)/k_2^2$$

and k_1, k_2, for every value of the applied load, β^2 load eccentricity, $\bar{e}$, and slenderness ratio, λ, are obtained from the simultaneous solution

of the following two nonlinear equations

$$A_{11}\sin k_1 + A_{13} + B_2(1) + \frac{k_2^2}{\lambda^2} = 0$$

$$A_{21}\sin k_2 + A_{23} - B_1(1) - \frac{k_1^2}{\lambda^2} = 0 \tag{2.23}$$

The expressions for U_T and joint rotation, φ, are

$$U_T = \frac{k_1^4 + k_2^4}{\lambda^2} + \frac{A_{11}^2 k_1^4}{2}\left(1 - \frac{\sin 2k_1}{2k_1}\right) + \frac{A_{21}^2 k_1^4}{2} - \left(1 - \frac{\sin 2k_2}{2k_2}\right)$$

$$+ 2\beta^2\left[\bar{e}(A_{21}k_2 \cos k_2 + A_{23}) - (A_{21}\sin k_2 + A_{23})\right] \tag{2.24}$$

$$\varphi = W_{1,X}(1) = A_{11}k_1 \cos k_1 + A_{13} \tag{2.25}$$

Model B

$$\Xi_1(X) = -\frac{k_1^2}{\lambda^2}X - B_1(X) \quad ;$$

$$W_1(X) = A_{11}\sin k_1 X + A_{13}X \quad ; \tag{2.26}$$

$$\Xi_2(X) = -\frac{k_2^2}{\lambda^2}X - B_2(X) \quad ; \quad W_2(X) = A_{21}\sin k_2 X + A_{24}$$

where

$$B_1(X) = \frac{1}{2}\left[A_{13}^2 X + 2A_{11}A_{13}\sin k_1 X + \frac{k_1^2 A_{11}^2}{2}\left(X + \frac{\sin 2k_1 X}{2k_1}\right)\right]$$

$$B_2(X) = \frac{k_2^2 A_{21}^2}{4}\left(X + \frac{\sin 2k_2 X}{2k_2}\right)$$

$$A_{11} = \frac{\beta^2 \bar{e} k_2 \cos k_2 + k_2^4 \sin k_2 / k_1^2}{k_1 k_2 (k_1 \sin k_1 \cos k_2 + k_2 \cos k_1 \sin k_2)} \tag{2.27}$$

$$A_{21} = \frac{\beta^2 \bar{e} k_1 \cos k_1 - k_2^2 \sin k_1}{k_1 k_2 (k_1 \sin k_1 \cos k_2 + k_2 \cos k_1 \sin k_2)}$$

$$A_{13} = -k_2^2/k_1^2 \quad ; \quad k_1^2 = \beta^2 \quad ; \quad A_{24} = B_1(1) + \frac{k_1^2}{\lambda^2} - A_{21} \sin k_2$$

and k_2(since for this case $k_1^2 = \beta^2$) for every value of the applied load, β^2, and structural geometry, $\bar{e}$, λ, can be found from the solution of the following nonlinear equation:

$$k_2^2 A_{21} \left[A_{21} \frac{\beta^2}{4} \left(1 + \frac{\sin 2k_2}{2k_2}\right) - \sin k_2 \right] + \bar{e}\beta^2 + k_2^2 \left(\frac{\beta^2}{\lambda^2} - 1\right) = 0 \tag{2.28}$$

The expressions for U_T and joint rotation, φ, are

$$U_T = \frac{k_1^4 + k_2^4}{\lambda^2} + \frac{A_{11}^2 k_1^4}{2} \left(1 - \frac{\sin 2k_1}{2k_1}\right) + \frac{A_{21}^2 k_2^4}{2} \left(1 - \frac{\sin 2k_2}{2k_2}\right) \tag{2.29}$$

$$+ 2\beta^2 \left[\bar{e}(A_{21} k_2 \cos k_2) - (A_{21} \sin k_2 + A_{24}) \right]$$

$$\varphi = A_{11} k_1 \cos k_1 + A_{13} \tag{2.30}$$

<u>Model C</u>

$$\Xi_1 = -\frac{k_1^2}{\lambda^2} X - B_1(X) \quad ; \quad W_1 = A_{11} \sin k_1 X$$

$$\Xi_2 = A_{25} - B_2(X) \quad ; \quad W_2 = A_{21} X^3 + A_{23} X \tag{2.31}$$

where

$$B_1(X) = \frac{k_1^2 A_{11}^2}{4}\left(X + \frac{\sin 2k_1 X}{2k_1}\right)$$

$$B_2(X) = \frac{1}{2}\left(\frac{9}{5} A_{21}^2 X^5 + 2A_{21}A_{23}X^3 + A_{23}^2 X\right)$$

$$A_{11} = \frac{k_1^2 + \beta^2(\bar{e} - 1)}{k_1^2 \sin k_1} \tag{2.32}$$

$$A_{21} = \frac{k_1^2 - \beta^2}{6}$$

$$A_{23} = \frac{k_1^2 + \beta^2(\bar{e} - 1)}{k_1} \cot k_1 + \frac{\beta^2 - k_1^2}{2}$$

$$A_{25} = A_{11} \sin k_1 + B_2(1) \quad ,$$

and k_1 is the solution of the following nonlinear equation (for any β^2, $\bar{e}$, and λ):

$$\frac{\beta^2 - k_1^2}{3} + \frac{k_1^2 + \beta^2(\bar{e} - 1)}{k_1} \cot k_1 - \frac{k_1^2}{\lambda^2}$$
$$- \frac{1}{4}\left[\frac{k_1^2 + \beta^2(\bar{e} - 1)}{k_1 \sin k_1}\right]^2 \left(1 + \frac{\sin 2k_1}{2k_1}\right) = 0 \tag{2.33}$$

Finally,

$$U_T = \frac{k_1^4}{\lambda^2} + \frac{A_{11}^2 k_1^4}{2}\left(1 - \frac{\sin 2k_1}{2k_1}\right) + 12A_{21}^2$$
$$+ 2\beta^2\left[\bar{e}\,(3A_{21} + A_{23}) - (A_{21} + A_{23})\right] \tag{2.34}$$

$$\varphi = A_{11}k_1 \cos k_1 \tag{2.35}$$

Numerical Results and Discussion

On the basis of the criterion established, critical loads are computed for all three frames and for a large practical range of load eccentricities ($-0.01 \le \bar{e} \le 0.01$) and of slenderness ratios (λ = 40, 80, ∞). The results are presented graphically in Figs. 2.3 - 2.5, and discussed separately for each frame (Model).

Model A: The results for this model are presented graphically on Fig. 2.3 and part of them in a tabular form on Table 2.1. It is observed that, as in the static case, there is a small positive eccentricity, $\bar{e}_{cr}$, such that for $\bar{e} \le \bar{e}_{cr}$ there is dynamic instability, while for $\bar{e} > \bar{e}_{cr}$ there is not. This $\bar{e}_{cr}$ is λ-dependent and identical to the corresponding static case. For all λ-values considered, except $\lambda \to \infty$, the difference between $\beta^2_{cr_D}$ and $\beta^2_{cr_{st}}$ is the largest at $\bar{e} = \bar{e}_{cr}$ and it diminishes as $\bar{e}$ increases negatively. On the contrary, for $\lambda \to \infty$ this effect is reversed and more specifically, the difference is close to zero at $\bar{e} = \bar{e}_{cr}$ and it increases as $\bar{e}$ increases negatively. In addition, eccentricity has a destabilizing effect regardless of the value of the slenderness ratio. This effect is less pronounced for the static case.

Finally, dynamic instability, as defined herein, takes place with a trajectory corresponding to a positive joint rotation φ. Because of this, of course, the compressive force in the vertical bar, k_1, is higher than the applied load, β^2, at the instant of possible "buckled" motion (trajectory possibly passing through the unstable static equilibrium point).

Note that the experimental results of Thompson (λ = 1275 [17] agree very well with the $\lambda \to \infty$ theoretical prediction. The largest discrepancy between theory and experiment is approximately 1.5%.

Model B: This is the only model, which exhibits bifurcational buckling (through an unstable branch) under static application of the load. The results are presented graphically in Fig. 2.4 and part of them in tabular form on Table 2.1.

It is seen from Fig. 2.4 that the effect of slenderness on the dynamic critical load is appreciable, while its effect on the static critical load (limit point load) is negligible. In addition, for all λ, except $\lambda \to \infty$, the difference between the static and dynamic critical loads is the largest at $\bar{e} = 0$ and decreases as $[\bar{e}]$ increases. Furthermore, at $\bar{e} = 0$ and for a given λ, except $\lambda \to \infty$, there are two dynamic critical loads, one corresponding to a negative rotation φ trajectory (the lower) and one corresponding to a positive φ trajectory (the upper). Definitely the system, for $\bar{e} = 0$, buckles in the mode associated with the lower load and it should be designed for this lower dynamic critical load. But the results indicate that a small positive eccentricity, in this case, has a stabilizing effect, because it forces the system to dynamically buckle through a positive rotation φ trajectory and therefore it can carry a higher load. In general, though, eccentricity has a destabilizing effect. This means that as $|\bar{e}|$ increases the dynamic critical load decreases.

Model C: The results for this model are presented graphically in Fig. 2.5 and part of them in tabular form on Table 2.1. The observations for this model are very similar to those corresponding to model A.

Table 21: Critical Conditions for λ = 80

Model	$\bar{c}$	k_1	k_2	$\beta^2_{cr_D}$	$\beta^2_{cr_D}/\beta^2_{cr_{st}}$
A	0.00047288	13.350303	0.701563	12.7054	0.915
	0.0000	13.319785	0.709696	12.6625	0.935
	-0.0013	13.239899	0.732511	12.5486	0.948
	-0.0025	13.170965	0.753974	12.4483	0.954
	-0.0050	13.040181	0.799120	12.2528	0.961
	-0.0070	12.946092	0.834976	12.1081	0.963
	-0.0100	12.819032	0.887372	11.9077	0.965
B	0	8.77116	.37756	8.77116	0.8887058
	.002	8.37378	.42278	8.37378	0.9010594
	.004	7.99803	.46167	7.99803	0.8986994
	.008	7.31164	.52393	7.31164	0.8867721
	.010	7.00021	.54887	7.00021	0.8804564
	0	9.24600	.31035	9.24600	0.9368172
	- .002	9.05455	.34513	9.05455	0.9554539
	- .004	8.88447	.37852	8.88447	0.9606390
	- .008	8.59210	.43900	8.59210	0.9592319
	- .010	8.46421	.46605	8.46421	0.9558657
C	0.0004722	1.37022	0	1.27144	0.895
	0.0000	1.36547	0	1.26421	0.921
	-0.0013	1.35332	0	1.24523	0.939
	-0.0025	1.34319	0	1.22885	0.946
	-0.0050	1.32475	0	1.19771	0.953
	-0.0075	1.30910	0	1.16995	0.955
	-0.0100	1.29550	0	1.14492	0.956

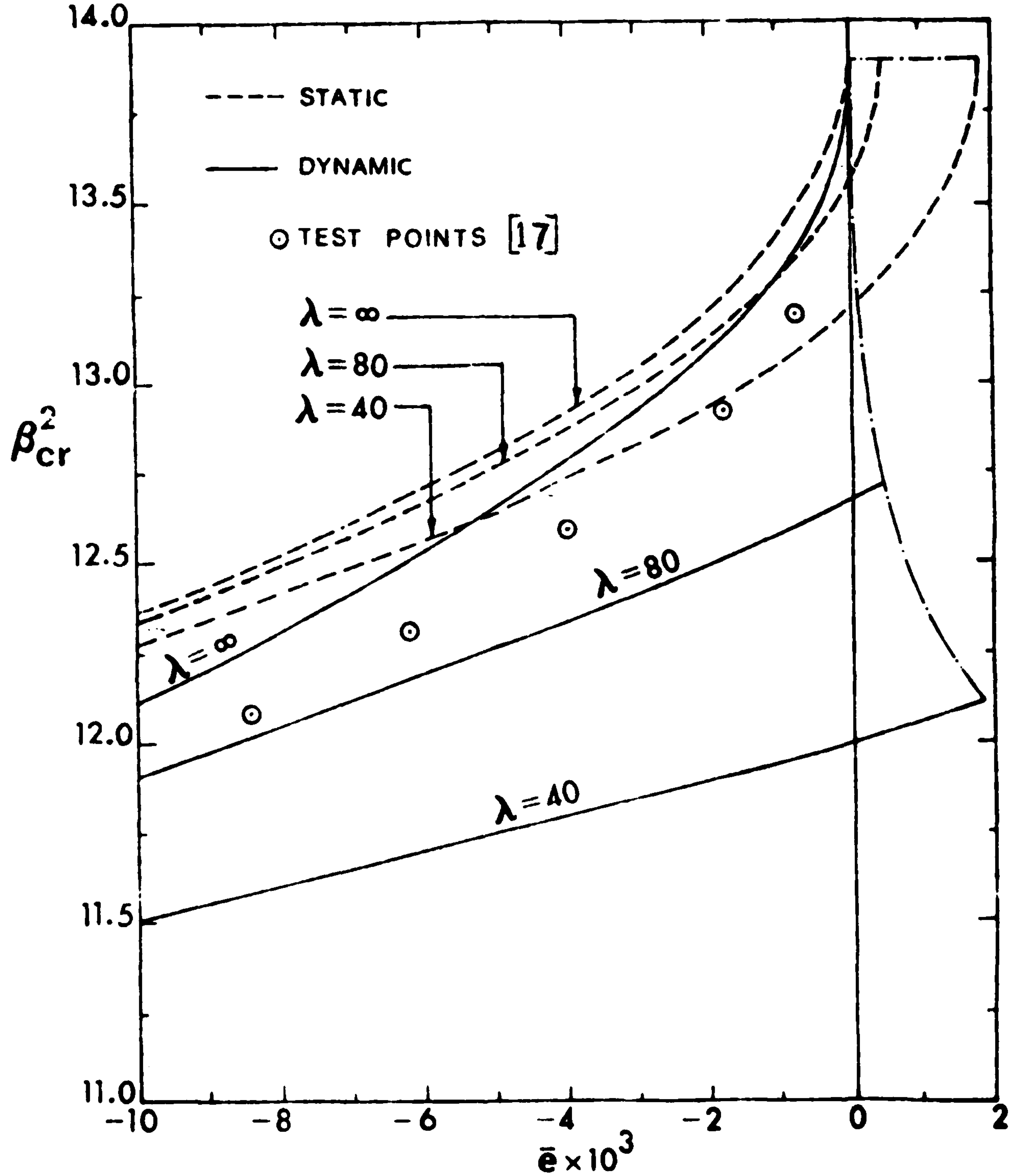

Fig. 2.3 Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model A)

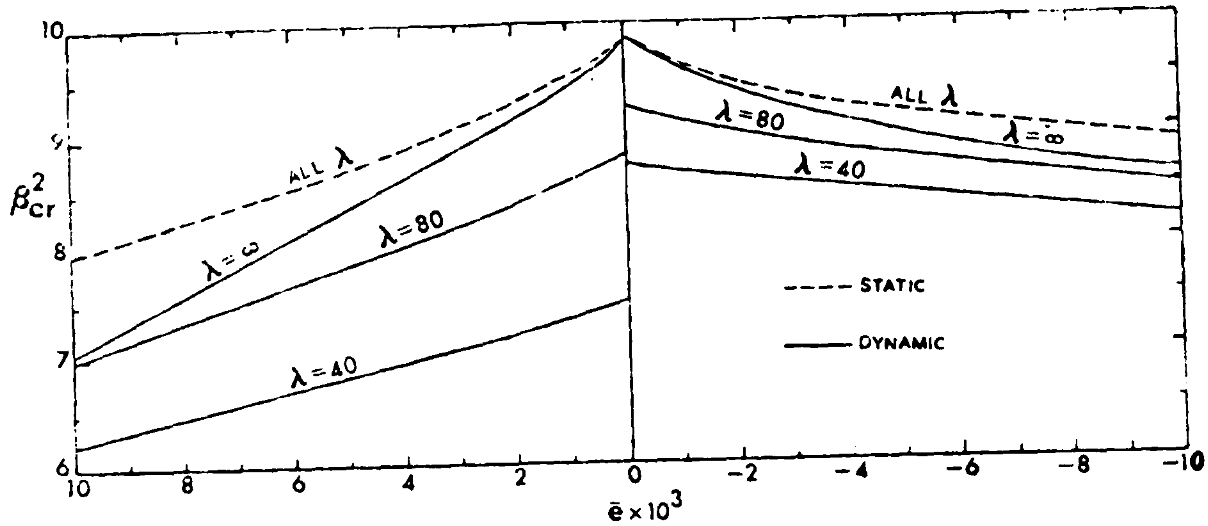

Fig. 2.4. Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model B).

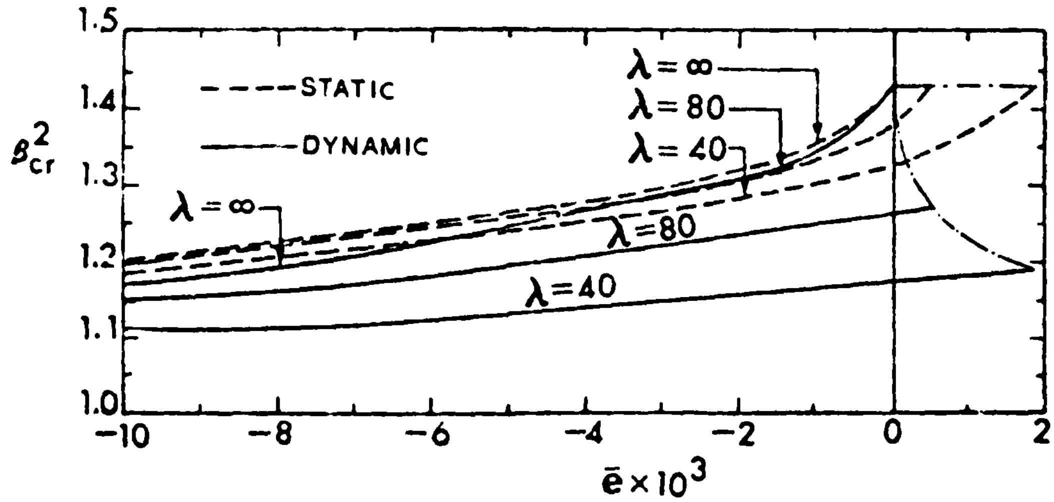

Fig. 2.5. Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model C).

In all three models, when the alternate method was employed, Eqs. (2.17) and (2.19), in some cases, the results were physically unacceptable. This is only mentioned here as a word of caution to those who may attempt to use this particular procedure for estimating dynamic critical loads. For the systems investigated herein, a less restrictive definition, related to the boundedness of the motion, may be employed, which is: an unbounded motion takes place for that magnitude of the applied load for which U_T assumes always negative values no matter what combination of kinematically admissible functions w_i, ξ_i can be assigned; Otherwise the motion is bounded.

SECTION III

STIFFENED AND UNSTIFFENED, IMPERFECT CYLINDRICAL SHELLS UNDER SUDDENLY APPLIED LOADS.

There are a few publications dealing with dynamic buckling of shell configurations. Some of them deal with shallow spherical caps (see [12] for a fairly complete review) and even fewer with cylindrical shells [4, 18-22], most of which are based on the Budiansky-Roth approach. In this chapter, the necessary criterion and the related solution methodology are presented, based on the energy approach [1].

The Stability Criterion

Consider a stiffened, geometrically imperfect, circular cylindrical shell, (see Fig. 3.1), supported in various ways (all possible boundary conditions) and loaded suddenly by a set of loads consisting of uniform axial compression and uniform pressure. These loads may be applied individually or in combination, but they will be, in general, represented by a load parameter λ. The case considered, herein, corresponds to suddenly applied loads of infinite duration and constant magnitude. Since the internal and external loads are conservative, the system is conservative.

Let u, v, and w be the reference surface displacement component (see next section) which are, in general, functions of position (x, y, z) and time, t. Then, the total potential is a functional of the displacement components and their space-dependent partial derivatives. Similarly, the kinetic energy is a functional (positive definite) of the time-dependent derivatives. The functional is said to be positive definite if it is positive for all possible values of the functions in the integrand, except zero, in which case the functional is zero.

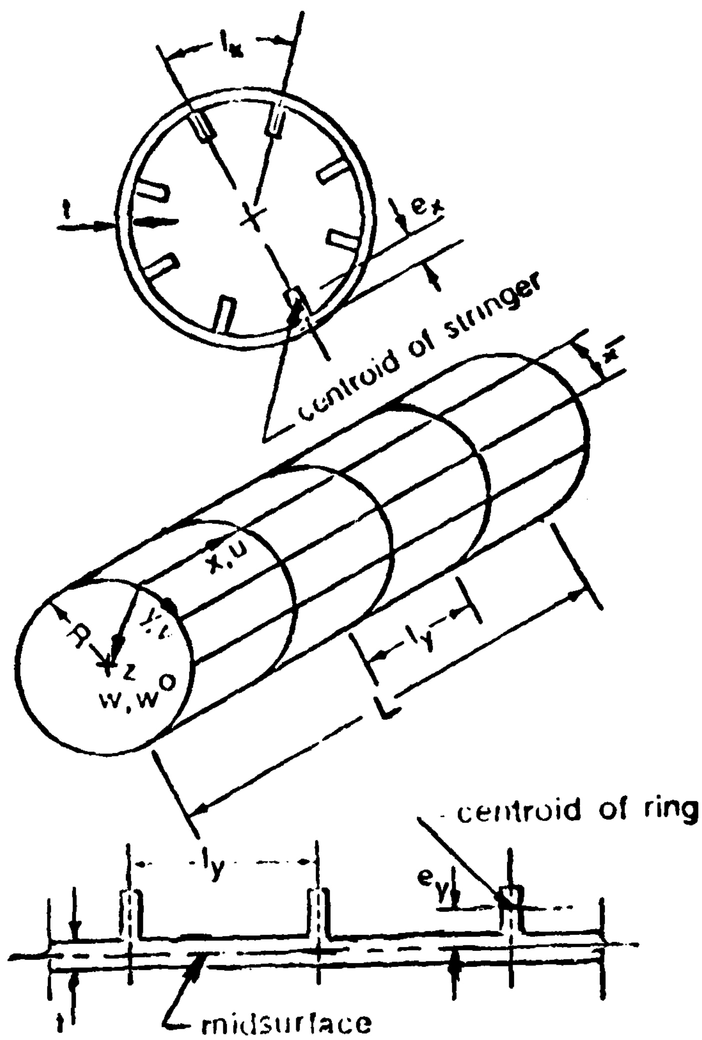

Fig. 3.1 Geometry and Sign Convention

On the basis of the conservation of energy principle, the total energy is a constant, C, or

$$U_T\left[u,v,w;\lambda\right] + T\left[u,_t,v,_t,w,_t\right] = C \tag{3.1}$$

where U_T and T are the total potential and kinetic energy functionals respectively and C is a constant as far as buckling trajectories are concerned. This means that, in the case of uniform pressure on a cylindrical shell, the breathing mode trajectory cannot possibly be considered as an admissible buckling trajectory. The reason for this lies in the fact that a perfect cylindrical shell with ends free to expand and/or contract when loaded by uniform pressure deforms in the breathing mode primarily (primary path), therefore "buckled" motion cannot possibly occur through this mode. Then, C in Eq. (3.1) will account for the potential of the external forces because of primary path modes. Note that in the case of axial compression C contains the potential of the axial force in connection with the axial mode (the part of u which is not w-dependent). This point is further dealt with in a later section.

Next, define a modified total potential, $U_{T_{mod.}}$ such that

$$U_{T_{mod.}} + T = 0 \tag{3.2}$$

where

$$U_{T_{mod.}} = U_T - C$$

With this modification the criterion becomes identical to that for the frame problem (Section 2).

The Static Solution

It is obvious from the stability criterion as applied to the case of sudden loads of constant magnitude and infinite duration, that a complete

static analysis is needed. This includes pre-limit point behavior, establishment of the limit point and post-limit point behavior. At each equilibrium point, the corresponding value of the modified total potential must be evaluated. According to the criterion, then, the value of the load for which the modified total potentia' in zero at an unstable equilibrium point (post-limit point) corresponds to a lower bound of the critical dynamic load.

Consider an imperfect, orthogonally stiffened, thin, circular, cylindrical shell (see Fig. 1), loaded by axial compression and/or uniform pressure. Let $w^{\circ}(x,y)$ denote the deviation of the shell midsurface (taken as a reference surface) from the corresponding perfectly cylindrical one. Moreover, let u, v, and w denote the displacement components of the material points on the reference surface. The component $w(x,y)$ is measured from the reference surface in the radial direction.

The nonlinear kinematic relations for this configuration are

$$\begin{aligned}
\epsilon^{\circ}_{xx} &= u,_x + \frac{1}{2}\left(w,_x^2 + 2w,_x w^{\circ},_x\right) \\
\epsilon^{\circ}_{yy} &= v,_y - \frac{w}{R} + \frac{1}{2}\left(w,_y^2 + 2w,_y w^{\circ},_y\right) \\
\gamma^{\circ}_{xy} &= 2\epsilon_{xy} = u,_y + v,_x + w,_x w,_y + w,_x w^{\circ},_y + w,_y w^{\circ},_x \\
\varkappa_{xx} &= w,_{xx} \; ; \; \varkappa_{yy} = w,_{yy} \; ; \; \varkappa_{xy} = w,_{xy}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\epsilon_{xx} &= \epsilon^{\circ}_{xx} - zw,_{xx} \\
\epsilon_{yy} &= \epsilon^{\circ}_{yy} - zw,_{yy} \\
\gamma_{xy} &= \gamma^{\circ}_{xy} - zzw,_{xy}
\end{aligned} \tag{3.4}$$

The relations of the stress and moment resultants to the strains and changes in curvature and torsion are (see Ref. 23):

$$
\begin{aligned}
N_{xx} &= E_{xx_p}\left[(1+\lambda_{xx})\epsilon^{\circ}_{xx} + \nu\epsilon^{\circ}_{yy} - e_x\lambda_{xx}\kappa_{xx}\right] \\
N_{yy} &= E_{xx_p}\left[\nu\epsilon^{\circ}_{xx} + (1+\lambda_{yy})\epsilon^{\circ}_{yy} - e_y\lambda_{yy}\kappa_{yy}\right] \\
N_{xy} &= E_{xx_p}\left[(1-\nu)\epsilon^{\circ}_{xy}\right] \\
M_{xx} &= D\left\{\left[(1+\rho_{xx}) + \frac{12}{t^2}e_x^2\lambda_{xx}\right]\kappa_{xx} + \nu\kappa_{yy} - \frac{12}{t^2}e_x\lambda_{xx}\epsilon^{\circ}_{xx}\right\} \\
M_{yy} &= D\left\{\nu\kappa_{xx} + \left[(1+\rho_{yy}) + \frac{12}{t^2}e_y^2\lambda_{yy}\right]\kappa_{yy} - \frac{12}{t^2}e_y\lambda_{yy}\epsilon^{\circ}_{yy}\right\} \\
M_{xy} &= D(1-\nu)\,\kappa_{xy}
\end{aligned}
\tag{3.5}
$$

where

$$E_{xx_p} = Et/(1-\nu^2); \quad D = Et^3/12(1-\nu^2); \quad \lambda_{xx} = A_x(1-\nu^2)/t\ell_x$$

$$\lambda_{yy} = A_y(1-\nu^2)/t\ell_y; \quad \rho_{xx} = EI_{xc}/D\ell_x; \quad \text{and } \rho_{yy} = EI_{yc}/D\ell_y.$$

From Eqs. (3.5) one may derive the following expressions for the reference surface strains

$$
\begin{aligned}
\epsilon^{o}_{xx} &= a_1N_{xx} + a_2N_{yy} + a_3\kappa_{xx} + a_4\kappa_{yy} \\
\epsilon^{o}_{yy} &= a_2N_{xx} + b_2N_{yy} + b_3\kappa_{xx} + b_4\kappa_{yy} \\
\epsilon^{o}_{xy} &= \frac{1}{2}\gamma_{xy} = N_{xy}/(1-\nu)E_{xx_p}
\end{aligned}
\tag{3.6}
$$

where

$$a_1 = (1+\lambda_{yy})/\alpha E_{xx_p} \; ; \quad a_2 = -\nu/\alpha E_{xx_p} \; ; \quad a_3 = (1+\lambda_{yy})e_x\lambda_{xx}/\alpha$$

$$a_4 = -\nu e_y\lambda_{yy}/\alpha \; ; \quad b_2 = (1+\lambda_{xx})/\alpha E_{xx_p} \; ; \quad b_3 = -\nu e_x\lambda_{xx}/\alpha \tag{3.7}$$

$$b_4 = (1+\lambda_{xx})e_y\lambda_{yy}/\alpha \; ; \quad \alpha = \left[(1+\lambda_{xx})(1+\lambda_{yy}) - \nu^2\right]$$

By employing the principle of the stationary value of the total potential one can derive the following equilibrium equations

$$N_{xx,x} + N_{xy,y} = 0$$

$$N_{xy,x} + N_{yy,y} = 0$$

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} = \frac{N_{yy}}{R} + \left[N_{yy}(w,_y+w^\circ,_y)\right],_y + \left[N_{xy}(w,_x+w^\circ,_x)\right],_y$$

$$+ \left[N_{xx}(w,_x+w^\circ,_x)\right],_x + \left[N_{xy}(w,_y+w^\circ,_y)\right],_x + p$$

By introducing the Airy stress function, as $N_{xx} = -\bar{N}_{xx} + F,_{yy}$, $N_{yy} = F,_{xx}$ and $N_{xy} = -F,_{xy}$ where $\bar{N}_{xx}$ is the level of the applied uniform axial compression, the first two of Eqs. (3.8) are identically satisfied.

Next, by eliminating u and v from the first three of Eqs. (3.3), employing Eqs. (3.6), the Airy stress function and the last three of Eqs. (3.3) one can derive the compatibility equation in terms of the Airy stress function, F and the radial displacement, w. If one expresses the third of Eq. (3.8) in terms of F and w, the governing equations consist of two coupled partial differential equations in F and w. These are:

Equilibrium

$$DL_h[w] - L_q[F] - F,_{xx}/R + \bar{N}_{xx}(w,_{xx} + w^o,_{xx}) - L[F,w + w^o] - p = 0 \quad (3.9)$$

Compatibility

$$L_d[F] + L_q[w] + \frac{1}{2}L\left[w,w + 2w^o\right] + w,_{xx}/R = 0 \quad (3.10)$$

where L_d, L_h, and L_q are differential operators defined by L_g,

$$L_g[S] = g_{11}S,_{xxxx} + 2g_{12}S,_{xxyy} + g_{22}S,_{yyyy} \quad (3.11)$$

with

$$\begin{aligned} d_{11} &= (1+\lambda_{xx})/\alpha E_{xx_p} \\ d_{12} &= [(1+\lambda_{xx})(1+\lambda_{yy}) - \nu]/\alpha(1-\nu)E_{xx_p} \\ d_{22} &= (1+\lambda_{yy})/\alpha E_{xx_p} \end{aligned} \quad (3.12)$$

$$\begin{aligned} h_{11} &= 1 + \rho_{xx} = \frac{12}{t^2} \cdot \frac{e_x^2\lambda_{xx}(1 + \lambda_{yy} - \nu^2)}{\alpha} \\ h_{12} &= 1 + \frac{12}{t^2}\frac{\nu e_x e_y \lambda_{xx}\lambda_{yy}}{\alpha} \\ h_{22} &= 1 + \rho_{yy} + \frac{12}{t^2}\frac{e_y^2\lambda_{yy}(1 + \lambda_{xx} - \nu^2)}{\alpha} \end{aligned} \quad (3.13)$$

$$\begin{aligned} q_{11} &= -\nu e_x\lambda_{xx}/\alpha \\ q_{12} &= [(1 + \lambda_{yy})e_x\lambda_{xx} + (1 + \lambda_{xx})e_y\lambda_{yy}]/(2\alpha) \\ q_{22} &= -\nu e_y\lambda_{yy}/\alpha \end{aligned} \quad (3.14)$$

and L is a differential operator defined by

$$L[S,T] = S,_{xx}T,_{yy} - 2S,_{xy}T,_{xy} + S,_{yy}T,_{xx} \tag{3.15}$$

The total potential expression, in terms of the Airy stress function and the radial displacement, is given below

$$\begin{aligned} U_T = & \frac{1}{2E_{xx_p}} \int_A (\beta_1 F,^2_{yy} + \beta_2 F,^2_{xx} + \beta_3 F,_{xx}F,_{yy} + \beta_4 F,^2_{xy})\, dA \\ & + \frac{D}{2} \int_A (\alpha_1 w,^2_{yy} + \alpha_2 w,^2_{xx} + \alpha_3 w,_{xx}w,_{yy} + \alpha_4 w,^2_{xy})\, dA - \int_A pwdA \\ & - \frac{\bar{N}_{xx}}{2E_{xx_p}} \int_A (2\beta_1 F,_{yy} + \beta_3 F,_{xx})\, dA + \frac{\beta_1}{E_{xx_p}} \pi RL\, \bar{N}^2_{xx} - \bar{N}_{xx} 2\pi RLe_{AV} \end{aligned} \tag{3.16}$$

where e_{AV} (average end shortening) is given by

$$e_{AV} = - \int_A u,_x\, dA/2\pi RL \tag{3.17}$$

and

$$\begin{aligned} & \beta_1 = d_{22}E_{xx_p}; \quad \beta_2 = d_{11}E_{xx_p}; \quad \beta_3 = -2\nu/\alpha; \quad \beta_4 = 2/(1-\nu) \\ & \alpha_1 = h_{22}; \quad \alpha_2 = h_{11}; \quad \alpha_3 = 2\nu \left[1 + \frac{12}{t^2} \frac{e_x e_y \lambda_{xx} \lambda_{yy}}{\alpha}\right] \alpha_4 = 2(1-\nu) \end{aligned} \tag{3.18}$$

Similarly, the expressions for the average end shortening and "unit end shortening" at y = 0 are given by

$$\begin{aligned} e_{AV} = & a_1 \bar{N}_{xx} - \frac{1}{2\pi RL} \int_0^{2\pi R} \int_0^L [a_1 F,_{yy} + a_2 F,_{xx} + a_3 w,_{xx} + a_4 w,_{yy} \\ & - \frac{1}{2} w,_x (w,_x + 2w^o,_x)]\, dxdy \end{aligned} \tag{3.19}$$

$$e = a_1\bar{N}_{xx} - \frac{1}{L}\int_0^L \Big[a_1F,_{yy} + a_2F,_{xx} + a_3w,_{xx} + a_4w,_{yy}$$
$$- \frac{1}{2}\, w,_x\,(w,_x + 2w^o,_x)\Big]_{y=0}\, dx \qquad (3.20)$$

Note that e measures the amount of end shortening per unit of cylinder length, L.

The associated boundary conditions are either kinematic or natural. Thus, one must prescribe the values of either u, v, w and $w,_x$ or N_{xx}, N_{xy}, Q^*_x and M_{xx}.

Before listing the various boundary conditions, the expressions for M_{xx} and Q^*_x in terms of F and w are given. Moreover, a few explanatory remarks are presented for certain boundary conditions.

First, the expressions for M_{xx} and Q^*_x are:

$$M_{xx} = \gamma_1 w,_{xx} + \gamma_2 w,_{yy} + \gamma_3(F,_{yy} - \bar{N}_{xx}) + \gamma_4 F,_{xx}$$
$$Q^*_x = (F,_{yy} - \bar{N}_{xx})(w,_x + w^o,_x) + F,_{xy}(w,_y + w^o,_y) - M_{xx},_x - 2M_{xy},_y \qquad (3.21)$$

where

$$\gamma_1 = Dh_{11}; \quad \gamma_2 = \frac{D}{2}\alpha_3; \quad \gamma_3 = -a_3; \quad \gamma_4 = -b_3 \qquad (3.22)$$

As far as the in-plane boundary condition at either end one may write

either	or
$N_{xx} = -\bar{N}_{xx}$	$\delta u = 0$ (u = prescribed)

(3.23)

but in terms of the Airy stress function one may write

either	or
$F,_{yy} = 0$	u = prescribed

(3.23a)

From the above it is clear that the true in-plane condition (at one of the two ends) must be $F,_{yy} = 0$. At the other end it could be $u = 0$. Note that if one assigns various values to u at a boundary, the corresponding load ($\bar{N}_{xx}$) is unknown. This approach is not covered in this report.

Finally, if the applied load, $\bar{N}_{xx}$, passes through the reference surface and the reference surface is hinged (for the simply supported case), then in this case one may write $M_{xx} = 0$. On the other hand if the load is applied in an eccentric manner then $M_{xx} \neq 0$ but $M_{xx} = \pm e\bar{N}_{xx}$, where e is the load eccentricity (if $e = 0$, one has the usual simply supported condition).

All possible (extreme) boundary conditions are included in the analysis and the related solution methodology (including the computer program). These are, simply supported (SS), free (FF) and clamped (CC) for all possible combinations of in-plane boundary conditions (i=1,2,3,4).

$$
\begin{aligned}
&\text{SS-i;} \quad w = 0; \quad M_{xx} = \pm e\bar{N}_{xx} \\
&\text{CC-i;} \quad w = w,_x = 0 \\
&\text{FF-i;} \quad Q_x^* = 0; \quad M_{xx} = \pm e\bar{N}_{xx}
\end{aligned}
\qquad
\begin{aligned}
&1.\quad F,_{xy} = F,_{yy} = 0 \\
&2.\quad F,_{xy} = 0; \quad u = 0 \\
&3.\quad v = F,_{yy} = 0 \\
&4.\quad v = 0; \; u = C \\
&\text{where } C = \text{constant}
\end{aligned}
\tag{3.24}
$$

The conditions in u and v can be expressed in terms of w and F as in Ref. 24. For example, the condition $u = C$ in SS -2 can be replaced by a condition expressed solely in terms of w, w^o, F and their gradients. This is accomplished by the following procedure:

This boundary condition, SS=2, at $x = 0$ or L is given by $w = 0$; $F,_{xy} = 0$, $M_{xx} = \pm e\bar{N}_{xx}$ and $u = C$. The first two are in terms of w and F. The third one, $M_{xx} = \pm e\bar{N}_{xx}$, from the first of Eqs. (3.21) is expressed in terms of w, F, and their gradients. For the last one, one

notes that [see Eqs. (3.3) and (3.5)]

$$\epsilon_{xy} = \frac{1}{2}\left[u,_y + v,_x + w,_x w,_y + w,_y w^o,_x + w,_x w^o,_y\right] = -F,_{xy}\big/(1-\nu)E_{xx_p} \quad (3.25)$$

since $F,_{xy} = 0$, $w,_y = 0$ because $w(0,y) = 0$, and $u,_y = 0$ because $u(0,y) = C$, Eq. (3.25) becomes

$$v,_x + w,_x w^o,_y = 0 \quad (3.26)$$

Similarly, from Eqs. (3.3) and (3.5) one may write

$$\begin{aligned} \epsilon_{yy} &= v,_y + \frac{1}{2}\left[w,_y(w,_y + 2w^o,_y)\right] - \frac{w}{R} \\ &= a_2 N_{xx} + b_2 N_{yy} + b_3 \kappa_{xx} + b_4 \kappa_{yy} \end{aligned} \quad (3.27)$$

This equation, Eq. (3.27), is valid at any point along the shell, therefore differentiation with respect to x does not violate its validity. If this is done and if the N's and κ's are expressed in terms of w, F, and their gradients, one may write

$$\begin{aligned} &v,_{yx} + \frac{1}{2}\left[w,_{xy}(w,_y + 2w^o,_y) + w,_y(w,_{xy} + 2w^o,_{xy})\right] - \frac{w,_x}{R} \\ &= a_2 F,_{yyx} + b_2 F,_{xxx} + b_3 w,_{xxx} + b_4 w,_{yyx} \end{aligned} \quad (3.28)$$

Evaluation of Eq. (3.28) at x = 0 or L, and use of the fact that $w,_y(0,y) = 0$ yields

$$v,_{yx} + w,_{xy} w^o,_y - w,_x/R = a_2 F,_{yyx} + b_2 F,_{xxy} + b_3 w,_{xxx} + b_4 w,_{yyx} \quad (3.29)$$

Differentiation of Eq. (3.26) with respect to y, yields

$$v,_{xy} + w,_{xy} w^o,_y + w,_x w^o,_{yy} = 0 \quad (3.30)$$

Substitution of Eq. (3.30) into Eq. (3.29) yields a boundary condition equivalent to u = C, or

$$b_2F,_{xxx} + b_3w,_{xxx} + b_4w,_{yyx} + w,_x\left(\frac{1}{R} + w^o,_{yy}\right) = 0 \tag{3.31}$$

Similar steps may be followed to express all possible boundary conditions in terms of w, F, and their gradients. In order to save space, only the final expression for all possible boundary conditions, Eqs. (3.24), are given below, which have been incorporated into the computer program (see Appendix A). The condition, shown below, corresponds to uniform application of $\bar{N}_{xx}$ across the cross-section ($M_{xx} = a_3\bar{N}_{xx}$).

$$\underline{\text{SS-1}}\quad w = \gamma_1w,_{xx} + \gamma_4F,_{xx} = F,_{xy} = F,_{yy} = 0$$

$$\underline{\text{SS-2}}\quad w = \gamma_1w,_{xx} + \gamma_3F,_{yy} + \gamma_4F,_{xx} = F,_{xy} = 0$$

$$b_2F,_{xxx} + b_3w,_{xxx} + b_4w,_{yyx} + w,_x\left(\frac{1}{R} + w^o,_{yy}\right) = 0$$

$$\underline{\text{SS-3}}\quad w = \gamma_1w,_{xx} + \gamma_4F,_{xx} = F,_{yy} = 0, b_2F,_{xx} + b_3w,_{xx} = a_2\bar{N}_{xx}$$

(3.32)

$$\underline{\text{SS-4}}\quad w = \gamma_1w,_{xx} + \gamma_3F,_{yy} + \gamma_4F,_{xx} = a_2\left(F,_{yy} - \bar{N}_{xx}\right) + b_2F,_{xx}$$

$$+ b_3w,_{xx} = 0$$

$$\left[a_2 + 2/(1-\nu)\dot{E}_{xx_p}\right]F,_{xyy} + b_2F,_{xxx} + b_3w,_{xxx} + b_4w,_{xyy}$$

$$+ w,_x\left(\frac{1}{R} + w^o,_{yy}\right) = 0$$

$$\underline{\text{CC-1}}\quad w = w,_x = F,_{xy} = F,_{yy} = 0$$

$$\underline{\text{CC-2}}\quad w = w,_x = F,_{xy} = 0 \qquad b_2F,_{xxx} + b_3w,_{xxx} = 0$$

$$\underline{\text{CC-3}}\quad w = w,_x = F,_{yy} = 0 \qquad b_2F,_{xx} + b_3w,_{xx} = a_2\bar{N}_{xx}$$

$$\underline{\text{CC-4}}\quad w = w,_x = a_2\left(F,_{yy} - \bar{N}_{xx}\right) + b_2F,_{xx} + b_3w,_{xx} = 0$$

$$\left[a_2 + 2/(1-\nu)E_{xx_p}\right] F,_{yyx} + b_2F,_{xxx} + b_3w,_{xxx} = 0 \tag{3.33}$$

Similarly the FF-1 condition and the symmetry and antisymmetry conditions at $x = L/2$ are

$$\underline{\text{FF-1}}\quad \gamma_1w,_{xx} + \gamma_2w,_{yy} + \gamma_4F,_{xx} = F,_{yy} = F,_{xy} = 0$$

$$\gamma_4F,_{xxx} + \gamma_1w,_{xxx} + \left[\gamma_2 + 2D(1-\nu)\right]w,_{xyy} + \bar{N}_{xx}(w,_x - w^o,_x) = 0 \tag{3.34}$$

Symmetry $\quad (w,_x = Q^* = N_{xy} = u = 0)$

$$w,_x = \gamma_1w,_{xxx} + \gamma_4F,_{xxx} - (F,_{yy} - \bar{N}_{xx})\, w^o,_x = 0 \tag{3.35}$$

$$F,_{xy} = b_2F,_{xxx} + b_3w,_{xxx} + w^o,_x w,_{yy} = 0$$

Antisymmetry $(w = M_{xx} = v = F,_{yy} = 0)$

$$w = \gamma_1w,_{xx} + \gamma_4F,_{xx} = 0 \tag{3.36}$$

$$F,_{yy} = 0,\ b_2F,_{xx} + b_3w,_{xx} = a_2\bar{N}_{xx}$$

For the case of zero load eccentricity the various boundary conditions become

$$\underline{\text{SS-1}}\quad w = \gamma_1w,_{xx} + \gamma_4F,_{xx} = F,_{xy} = F,_{yy} = 0$$

$$\underline{\text{SS-2}}\quad w = \gamma_1w,_{xx} + \gamma_3\left(F,_{yy} - \bar{N}_{xx}\right) + \gamma_4F,_{xx} = F,_{xy} = 0$$

$$b_2F,_{xxx} + b_3w,_{xxx} + b_4w,_{yyx} + w,_x\left(\frac{1}{R} + w^o,_{yy}\right) = 0$$

$$\underline{\text{SS-3}}\quad w = \gamma_1w,_{xx} + \gamma_4F,_{xx} = F,_{yy} = b_2F,_{xx} + b_3w,_{xx} = a_2\bar{N}_{xx}$$

$$\underline{\text{SS-4}} \quad w = \gamma_1 w,_{xx} + \gamma_3\left(F,_{yy} - \bar{N}_{xx}\right) + \gamma_4 F,_{xx} = a_2\left(F,_{yy} - \bar{N}_{xx}\right)$$

$$+ b_2 F,_{xx} + b_3 w,_{xx} = 0$$

$$\left[a_2 + 2/(1-\nu)E_{xx_p}\right] F,_{xyy} + b_2 F,_{xxx} + b_3 w,_{xxx} + b_4 w,_{xyy} \tag{3.37}$$

$$+ w,_x \left(\frac{1}{R} + w^o,_{yy}\right) = 0$$

$$\underline{\text{CC-1}} \quad w = w,_x = F,_{xy} = F,_{yy} = 0$$

$$\underline{\text{CC-2}} \quad w = w,_x = F,_{xy} = 0 \qquad b_2 F,_{xxx} + b_3 w,_{xxx} = 0$$

$$\underline{\text{CC-3}} \quad w = w,_x = F,_{yy} = 0 \qquad b_2 F,_{xx} + b_3 w,_{xx} = a_2 \bar{N}_{xx} \tag{3.38}$$

$$\underline{\text{CC-4}} \quad w = w,_x = a_2\left(F,_{yy} - \bar{N}_{xx}\right) + b_2 F,_{xx} + b_3 w,_{xx} = 0$$

$$\left[a_2 = 2/(1-\nu)E_{xx_p}\right] F,_{yyx} + b_2 F,_{xxx} + b_3 w,_{xxx} = 0$$

Similarly the FF-1 conditions are

$$\underline{\text{FF-1}} \quad \gamma_1 w,_{xx} + \gamma_2 w,_{yy} + \gamma_4 F,_{xx} = F,_{yy} = F,_{xy} = 0$$

$$\gamma_4 F,_{xxx} + \gamma_1 w,_{xxx} + \left[\gamma_2 + 2D(1-\nu)\right] w,_{xyy} + N_{xx}\left(w,_x - w^o,_x\right) = 0 \tag{3.39}$$

The problem, as formulated herein, is to find the complete nonlinear response of the shell to externally applied pressure and compression. This response includes post-limit point behavior or postbuckling behavior, whichever is applicable. Thus, all equilibrium positions may be presented as plots of applied load parameter versus some characteristic displacement (the average end shortening is one possibility). Moreover, at each equilibrium point, the values of the total potential and modified total potential are recorded, in order to establish critical dynamic conditions.

Solution Methodology

The solution methodology described, herein, is an extension of the procedure outlined in [25]. Therefore, some duplication is unavoidable, especially in the interest of making this section self-contained.

As seen from Eqs. (3.9) and (3.10), the field equations consist of two coupled, nonlinear, partial differential equations in terms of the transverse displacement component, w, and the Airy stress function F.

A separated solution of the form shown below (see [24]), is used in order to reduce the system of partial differential equations to one of ordinary differential equations.

$$w(x,y) = \sum_{i=0}^{K} W_i(x) \cos \frac{iny}{R}$$

$$F(x,y) = \sum_{i=0}^{2K} f_i(x) \cos \frac{iny}{R} \tag{3.40}$$

where n denotes the number of full waves around the circumference.

The initial geometric imperfection is also expressed in a similar form

$$W^o(x,y) = \sum_{i=0}^{K} W_i^o(x) \cos \frac{iny}{R} \tag{3.41}$$

where $W_i^o(x)$ denotes known functions of position x.

The following steps are employed in order to accomplish the reduction to a system of ordinary differential equations. First, Eqs. (3.40) and (3.41) are substituted into the compatibility equation [Eq. (3.10)]. Next, by employing trigonometric identities of double Fourier series involving products (as in [26]), the compatibility equation reduces to a trigonometric

series in $\frac{iny}{R}$. The coefficient of each term involves differential operations on W_i, f_i, and W_i^o. Use of the orthogonality of the trigonometric functions reduces the compatibility equation into 2K + 1 ordinary, nonlinear, differential equations.

Next, the Galerkin procedure is employed (in the circumferential direction) in connection with the equilibrium equation [Eq. (3.9)]. This leads to the vanishing of (K + 1) Galerkin integrals, which results into a system of (K + 1) nonlinear ordinary differential equations in W_i and f_i.

These equations are:

(a) Compatibility (2K + 1)

<u>for i = 0</u>

$$f_o'' = \frac{1}{d_{11}}\left[-q_{11}W_o'' - W_o/R + \frac{n^2}{4R^2}\sum_{j=1}^{K} j^2(W_j + 2W_j^o)W_j + a_2\bar{N}_{xx}\right] \qquad (3.42)$$

The above equation is obtained from the first (fourth order) compatibility equation, with the continuity condition on v at zero and 2π satisfied.

<u>for i = 1, 2,...,2K</u>

$$\begin{aligned} d_{11}f_i'''' &- 2\left(\frac{in}{R}\right)^2 d_{12}f_i'' + \left(\frac{in}{R}\right)^4 d_{22}f_i \\ &+ \delta_i\left[q_{11}W_i'''' - 2\left(\frac{in}{R}\right)^2 q_{12}W_i'' + \left(\frac{in}{R}\right)^4 q_{22}W_i + W_i''/R\right] \\ &- \frac{n^2}{4R^2}\sum_{j=0}^{K}\Big\{\Big[(i+j)^2\delta_{i+j}(W_{i+j} + 2W_{i+j}^o) \qquad (3.43) \\ &+ (2 - \eta_{j-i}^2)(i-j)^2\delta_{|i-j|}(W_{|i-j|} + 2W_{|i-j|}^o)\Big]W_j'' \\ &+ [\delta_{i+j}(W_{i+j}'' + 2W_{i+j}^{o''}) + (2 - \eta_{j-i}^2)\delta_{|i-j|}(W_{|i-j|}'' \end{aligned}$$

$$+ 2W^{o''}_{|i-j|})]j^2W_j + 2[(i+j)\delta_{i+j}(W'_{i+j} + 2W^{o'}_{i+j})$$

$$- \eta_{i-j}|i-j|\delta_{|i-j|}(W'_{|i-j|} + 2W^{o'}_{|i-j|})]jW'_j\} = 0$$

where

$$\delta_\ell = \begin{cases} 0 & \ell > K \\ 1 & \ell \le K \end{cases} \qquad \eta_\ell = \begin{cases} -1 & \ell < 0 \\ 0 & \ell = 0 \\ 1 & \ell > 0 \end{cases}$$

and

$$(\ )' = \frac{d}{dx}\ .$$

(b) Equilibrium (K + 1)

<u>for i = 0</u>

$$W_o''''[Dh_{11} + q_{11}^2/d_{11}] + W_o''[2q_{11}/R\cdot d_{11}] + W_o[1/R^2\cdot d_{11}] + \bar{N}_{xx}(W_o'' + W_o^{o''})$$

$$- \frac{n^2}{4R^2}\sum_{j=1}^{K} j^2\left\{\frac{q_{11}}{d_{11}}[(W_j + 2W_j^o)W_j'' + (W_j'' + 2W_j^{o''})W_j\right.$$

$$+ 2(W_j' + 2W_j^{o'})W_j'] + \frac{1}{Rd_{11}}[(W_j + 2W_j^o)W_j] - 2[(W_j + W_j^o)f_j'' \tag{3.44}$$

$$\left. + (W_j'' + W_j^{o''})f_j + 2(W_j' + W_j^{o'})f_j']\right\} - \frac{a_2\bar{N}_{xx}}{Rd_{11}} = 0$$

<u>for i = 1, 2,...,K</u>

$$D\left[h_{11}W_i'''' - 2\left(\frac{in}{R}\right)^2 h_{12}W_i'' + \left(\frac{in}{R}\right)^4 h_{22}W_i\right]$$

$$- \left[q_{11}f_i'''' - 2\left(\frac{in}{R}\right)^2 q_{12}f_i'' + \left(\frac{in}{R}\right)^4 q_{22}f_i + f_i''/R\right]$$

$$+ \bar{N}_{xx} (W_i'' + W_i^{o''}) - W_o'' \left[\frac{q_{11}}{d_{11}} \left(\frac{in}{R}\right)^2\right] (W_i + W_i^o) - W_o \left[\frac{1}{Rd_{11}} \left(\frac{in}{R}\right)^2 (W_i\right.$$

$$+ W_i^o) + \frac{n^4}{4R^4} \frac{i^2}{d_{11}} (W_i + W_i^o) \sum_{j=1}^{K} j^2 (W_j + 2W_j^o) W_j$$

$$+ \frac{n^2 i^2}{R^2} (W_i + W_i^o) \frac{a_2 \bar{N}_{xx}}{d_{11}} + \frac{n^2}{2R^2} \sum_{j=1}^{2K} \{ [(i+j)^2 \delta_{i+j} (W_{i+j} + W_{i+j}^o)$$

$$+ (2 - \eta_{j-i}^2)(i-j)^2 \delta_{|i-j|} (W_{|i-j|} + W^o_{|i-j|})] f_j'' \tag{3.45}$$

$$+ [\delta_{i+j} (W_{i+j}'' + W_{i+j}^{o''}) + (2 - \eta_{j-i}^2) \delta_{|i-j|} (W_{|i-j|}'' + W_{|i-j|}^{o''})] j^2 f_j$$

$$+ 2[(i+j) \delta_{i+j} (W_{i+j}' + W_{i+j}^{o'})$$

$$- \eta_{i-j} |i-j| \delta_{|i-j|} (W_{|i-j|}' + W_{|i-j|}^{o'})] j f_j' \} = 0$$

For a given imperfection and value of the applied load, $\bar{N}_{xx}$, Eqs. (3.42)-(3.45) represent a system of (3K + 2) coupled nonlinear differential equations in (3K + 2) unknowns, f_i with i = 0, 1, 2,...,2K and W_i with i = 0, 1, 2,...,K.

Note that by setting n=0, Eqs. (3.42)-(3.45) reduce to the linearized version of the equations of compatibility and equilibrium. Moreover, it is seen from Eqs. (3.40) that regardless of the value of n(=1,2,...any integer) the axisymmetric mode (W_o, f_o) is represented because the summation on $\underline{i}$ starts from zero.

In addition, it is seen from Eq. (3.41) that the imperfection expression is suitable for the case when the imperfection shape is similar to the buckling mode, as well as for any arbitrary axisymmetric imperfection and for any arbitrary symmetric (with respect to y) imperfection.

In this last case, a solution can be accomplished by setting n=1, [see Eqs. (3.40)], and by taking K sufficiently large in order to achieve a convergent solution and have an accurate representation for the imperfection.

Next, the boundary conditions (at x = constant), for two cases of simple supports are presented below.

<u>SS-1</u> ($M_{xx} = 0$)

$$W_o = 0 \; ; W_o'' = -\gamma_4 a_2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' + \gamma_4 f_i'' = 0; \quad i = 1, 2, 3, \ldots, K \tag{3.46a}$$

$$f_i = f_i' = 0; \quad i = 1, 2, 3, \ldots, 2K$$

<u>SS-1</u> ($M_{xx} = a_3 \bar{N}_{xx}$)

$$W_o = 0; \quad W_o'' = -\gamma_4 a_2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' + \gamma_4 f_i'' = 0; \quad i = 1, 2, 3, \ldots, K \tag{3.46b}$$

$$f_i = f_i' = 0; \quad i = 1, 2, 3, \ldots, 2K$$

<u>SS-2</u> ($M_{xx} = 0$)

$$W_o = 0; \; W_o'' = -\gamma_4 a_2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11}) + \gamma_3 \bar{N}_{xx}$$

$$W_i = \gamma_1 W_i'' - \gamma_3 \left[\bar{N}_{xx} + \left(\frac{in}{R}\right)^2 f_i\right] + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K$$

$$f_i' = 0; \quad i = 1, 2, 3, \ldots, 2K$$

$$b_2 f_i''' + b_3 W_i''' - \left(\frac{in}{R}\right)^2 b_4 W_i' + \frac{1}{R} W_i' - \frac{n^2}{2R^2} \sum_{j=0}^{K} \Big[(i+j)^2 W^o_{i+j} \tag{3.47a}$$

$$+ (1 - \eta^2_{j-i} + \eta_i)\,(i-j)^2 W^o_{|i-j|} \Big] W_j' = 0; \quad i = 1, 2, \ldots, 2K$$

<u>SS-2</u> $(M_x = a_3 \overline{N}_{xx})$

$$W_o = 0; \quad W_o'' = -\gamma_4 a_2 \overline{N}_{xx} / (\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' - \gamma_3 \left(\frac{in}{R}\right)^2 f_i + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K$$

$$f_i' = 0; \quad i = 1, 2, 3, \ldots, 2K \tag{3.47b}$$

$$b_2 f_i''' + b_3 W_i''' - \left(\frac{in}{R}\right)^2 b_4 W_i' + \frac{1}{R} W_i' - \frac{n^2}{2R^2} \sum_{j=0}^{K} \Big[(i+j)^2 W^o_{i+j}$$

$$+ (1 - \eta^2_{j-i} + \eta_i)\,(i-j)^2 W^o_{|i-j|} \Big] W_j' = 0; \quad i = 1, 2, \ldots, 2K$$

<u>SS-3</u> $(M_{xx} = 0)$

$$W_o = 0; \; W_o'' = -\gamma_4 a_2 \overline{N}_{xx} / (\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K \tag{3.48a}$$

$$f_i = b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$$

SS-3 $(M_{xx} = a_3\bar{N}_{xx})$

$$W_o = 0; \quad W_o'' = -\gamma_4 a_2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K \tag{3.48b}$$

$$f_i = b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$$

SS-4 $(M_{xx} = 0)$

$$W_o = 0; \; W_o'' = -\gamma_4 a_2^2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11}) + \gamma_3 \bar{N}_{xx}$$

$$W_i = \gamma_1 W_i'' - \gamma_3\left[\bar{N}_{xx} + \left(\frac{in}{R}\right)^2 f_i\right] + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K$$

$$- a_2\left[\bar{N}_{xx} + \left(\frac{in}{R}\right)^2 f_i\right] + b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$$

$$- \left[a_2 + 2/(1-\nu)E_{xx_p}\right]\left(\frac{in}{R}\right)^2 f_i' + b_2 f_i''' + b_3 W_i''' \tag{3.49a}$$

$$- b_4\left(\frac{in}{R}\right)^2 W_i' + \frac{1}{R} W_i' - \frac{n^2}{2R^2} \sum_{j=0}^{K}\left[(i+j)^2 \, W_{i+j}^o + (1-\eta_{j-i}^2\right.$$

$$\left. + \eta_i) \, W_{|i-j|}^o\right] W_j' = 0; \quad i = 1, 2, \ldots, 2K$$

SS-4 $(M_{xx} = a_3\bar{N}_{xx})$

$$W_o = 0; \quad W_o'' = -\gamma_4 a_2 \bar{N}_{xx}/(\gamma_1 d_{11} - \gamma_4 q_{11})$$

$$W_i = \gamma_1 W_i'' - \gamma_3\left(\frac{in}{R}\right)^2 f_i + \gamma_4 f_i'' = 0; \quad i = 1, 2, \ldots, K$$

$$- a_2\left(\frac{in}{R}\right)^2 f_i + b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$$

$$- \left[a_2 + 2/(1-\nu)E_{xx_p}\right]\left(\frac{in}{R}\right)^2 f_i' + b_2 f_i''' + b_3 W_i''' - b_4\left(\frac{in}{R}\right)^2 W_i'$$

$$+ \frac{1}{R} W_i' - \frac{n^2}{2R^2} \sum_{j=0}^{K} \left[(i+j)^2 W_{i+j}^o + (1 - \eta_{j-i}^2 \right. \tag{3.49b}$$

$$\left. + \eta_i) W_{|i-j|}^o \right] W_j' = 0; \quad i = 1, 2, \ldots, 2K$$

<u>CC-j</u> $\quad W_i = W_i' = 0; \quad i = 0, 1, \ldots, K$ (3.50)

(j=1,2,3,4)

and

<u>CC-1</u> $\quad f_i = f_i' = 0; \quad i = 1, 2, \ldots, 2K$ (3.50a)

<u>CC-2</u> $\quad f_i' = b_2 f_i''' + b_3 W_i''' = 0; \quad i = 1, 2, \ldots, 2K$ (3.50b)

<u>CC-3</u> $\quad f_i = b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$ (3.50c)

<u>CC-4</u> $\quad - a_2\left(\frac{in}{R}\right)^2 f_i + b_2 f_i'' + b_3 W_i'' = 0; \quad i = 1, 2, \ldots, 2K$

$$- \left[a_2 + 2/(1-\nu)E_{xx_p}\right]\left(\frac{in}{R}\right)^2 f_i' + b_2 f_i''' + b_3 W_i''' = 0, \tag{3.50d}$$

$i = 1, 2, \ldots, 2K$

Note that Eq. (3.42) is employed to eliminate f_o'' from the remaining equations, and there are no boundary conditions with reference to (f_o''). Thus, the number of boundary conditions, at the x = constant boundaries, is equal to (6K + 2) instead of (6K + 4).

Similarly, the expressions for the total potential, U_T, average end shortening, e_{AV}, and "unit end shortening", e, can be written in terms of f_i ($i = 1, 2, \ldots, 2K$) and W_i ($i = 0, 1, 2, \ldots, K$).

$$
\begin{aligned}
U_T = \pi R \int_0^L \Bigg[& \frac{1}{E_{xx_p}} \left\{ \frac{\beta_2}{d_{11}^2} \left[-\frac{W_o}{R} - q_{11} W_o'' + a_2 \bar{N}_{xx} \right.\right. \\
& \left. + \frac{n^2}{4R^2} \sum_{i=1}^{K} i^2 \left(W_i + 2W_i^o \right) W_i \right]^2 + \frac{1}{2} \sum_{i=1}^{2K} \left[\beta_1 \left(\frac{in}{R} \right)^4 f_i^2 \right. \\
& \left.\left. + \beta_2 f_i''^2 - \beta_3 \left(\frac{in}{R} \right)^2 f_i f_i'' + \beta_4 \left(\frac{in}{R} \right)^2 f_i'^2 \right] \right\} + D \left\{ \alpha_2 W_o''^2 \right. \\
& + \frac{1}{2} \sum_{i=1}^{K} \left[\alpha_1 \left(\frac{in}{R} \right)^4 W_i^2 + \alpha_2 W_i''^2 - \alpha_3 \left(\frac{in}{R} \right)^2 W_i'' W_i \right. \\
& \left.\left. + \alpha_4 \left(\frac{in}{R} \right)^2 W_i'^2 \right] \right\} - \bar{N}_{xx} \left\{ W_o' \left(W_o' + 2W_o^{o\prime} \right) \right. \\
& \left. + \frac{1}{2} \sum_{i=1}^{K} \left[W_i' \left(W_i' + 2W_i^{o\prime} \right) \right] \right\} \Bigg] dx \\
& + 2\pi R a_3 \bar{N}_{xx} \int_0^L W_o'' \, dx - \bar{N}_{xx}^2 \frac{\pi R L \beta_1}{E_{xx_p}}
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
e_{AV} = a_1 \bar{N}_{xx} + \frac{1}{L} \int_0^L \Bigg\{ & \frac{a_2}{d_{11}} \left[\frac{W_o}{R} + q_{11} W_o'' + a_2 \bar{N}_{xx} \right. \\
& \left. - \frac{n^2}{4R^2} \sum_{i=1}^{K} i^2 \left(W_i + 2W_i^o \right) W_i \right] - a_3 W_o'' \\
& + \frac{1}{2} W_o' \left(W_o' + W_o^{o\prime} \right) + \frac{1}{4} \sum_{i=1}^{K} W_i' \left(W_i' + W_i^{o\prime} \right) \Bigg\} \, dx
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
e = a_1 N_{xx} + \frac{1}{L}\int_0^L \Bigg[& \frac{a_2}{d_{11}} \bigg[\frac{W_o}{R} + q_{11} W_o'' + a_2 \bar{N}_{xx} \\
& - \frac{n^2}{4R^2} \sum_{i=1}^{K} i^2 (W_i + 2W_i^o) W_i \bigg] + \sum_{i=1}^{2K} \bigg[q_1 \left(\frac{in}{R}\right)^2 f_i \\
& - a_2 f_i'' \bigg] - a_3 \sum_{i=0}^{K} W_i'' + a_4 \sum_{i=1}^{K} \left(\frac{in}{R}\right)^2 W_i \\
& + \frac{1}{2} \bigg[\sum_{i=0}^{K} W_i' \bigg] \sum_{i=0}^{K} (W_i' + 2W_i^{o\prime}) \Bigg] dx
\end{aligned}
\tag{3.53}
$$

The solution methodology employed is described below, and it involves two solution schemes, one for finding equilibrium positions up to the limit point and one past the limit point.

First, a generalization of Newton's method [27,28], applicable to differential equations, is employed to reduce the nonlinear field equations, Eqs. (3.42)-(3.45) and appropriate boundary conditions to a sequence of linearized systems. In this method, the iteration equations (linearized system) are derived by assuming that the solution can be achieved by a small correction to an approximate solution.

For finding pre-limit point equilibrium positions, the applied load level, $\bar{N}_{xx}$, is taken as known, the linear (n≡0) solution is taken to be the approximate solution, and the small corrections (in W_i's, and f_i's) are obtained through the solution of the linearized (with respect to the corrections) differential equations. Note that, in this range, the stiffness matrix is positive definite.

For finding post-limit point equilibrium positions (in a range of negative stiffness matrix), the numerical scheme is modified. The load

parameter, $\overline{N}_{xx}$, is taken to be unknown, and one of the displacement parameters W_i replaces it as a known parameter. Great care must be exercised in choosing this W_i. This is done by observing how the various W_i's change with $\overline{N}_{xx}$ changes in the pre-limit point range, and choosing a W_i that tends to increase in a smooth and continuous manner, but most importantly is one of the most dominant displacement terms. In this post-limit point range, the linear solution cannot be taken as the initial estimate for the needed iterations. Therefore, the last converged, pre-limit point solution is used as an initial estimate for finding the first post-limit point solution. From there on, in this same range, the previous solution is utilized as an initial estimate. Needless to say that this latter procedure may also be used in the pre-limit point range, starting near the undeformed position. Unfortunately, this procedure is not very economical with regard to computer time and, therefore, it is very inefficient in this range.

It is decided to increase the number of dependent variables from $(3K + 1)$ $(W_o, W_1, \ldots W_K, f_1, f_{2K})$ to $(6K + 2)$ $(W_o, W_1, \ldots W_K, f_1 \ldots, f_{2K}, \eta_o, \eta_1, \ldots \eta_K, \xi_1, \ldots, \xi_{2K})$

where

$$\eta_i = W_i'' \qquad i = 0, 1, 2, \ldots, K$$

and (3.54)

$$\xi_j = f_j'' \qquad j = 1, 2, \ldots, 2K$$

The reason for this reduction of the order of the field equations, but increase of the number of the field equations, is related to the solution scheme, which is based on the finite difference procedure. In finite differences, it is convenient to keep the order of differential equations as low as possible (first and second order preferably). Then, the

linearized (in the increments) field equations, Eqs. (3.43)-(3.45), the transformation equations, Eqs. (3.54), and boundary terms, can be written in matrix form, as shown below:

Field Equations

$$[R]\,[Z''] + [S]\,\{Z'\} + [T]\,\{Z\} = \{g\} \tag{3.55}$$

Boundary Terms

$$[\bar{S}]\,\{Z'\} + [\bar{T}]\,\{Z\} = \{\bar{g}\} \tag{3.56}$$

where $\{Z\}$ is the vector of the (6K + 2) unknowns. Note that when the load parameter is considered as a known term, then

$$\{Z\}^T = \{W_o, W_1, \ldots, W_K, f_1, f_2, \ldots, f_{2K}, \eta_o, \eta_1, \ldots, \eta_K, \xi_1, \xi_2, \ldots, \xi_{2K}\} \tag{3.57}$$

On the other hand, if a certain W_j (chosen dominant term) is considered as a known term, this W_j is removed from vector $\{Z\}$, Eq. (3.61), and it is replaced by the load parameter.

Also, note that $[R]$, $[S]$, $[T]$, $[\bar{S}]$, $[\bar{T}]$, $\{g\}$, and $\{\bar{g}\}$ in Eqs. (3.55) and (3.56) contain known terms (associated with initial approximate solution and applied known increments). The ordinary differential equations are next cast into the form of finite difference equations. Thus, the linear differential equations, Eqs. (3.55) and (3.56) are changed into a system of linear algebraic equations. The usual central difference formula is used at all mesh points, ℓ, i.e.,

$$\begin{aligned} Z'_\ell &= (Z_{\ell+1} - Z_{\ell-1})/2\Delta \\ Z''_\ell &= (Z_{\ell+1} - 2Z_\ell + Z_{\ell-1})/\Delta^2 \end{aligned} \tag{3.58}$$

Note that, since the second derivatives in W_i and f_i are taken as independent variables, Eqs. (3.54), the second of Eqs. (3.58) is only applied to the fourth derivatives of W_i and f_i.

By using one fictitious point outside the cylinder at each end, one obtains a system of (6K + 2) X (NP + 2) linear difference equations. (Note that NP stands for number of mesh points.) These equations are solved by an algorithm which is a modification of the one described in [29]. A computer program has been written for the Georgia Tech high speed digital computer CDC-CYBER-70, Model 74-28. The listing and flow chart are given in the Appendices B and A respectively.

In generating data, in order to investigate pre- and post-limit point behavioral response of axially loaded cylindrical shells, the solution procedure goes as follows: first, the system of equations is solved for a small level of the applied load, $\bar{N}_{xx}$, (taken as known). Then, solutions are sought for step increases in $\bar{N}_{xx}$, until the process fails to converge. The load level at which the solution fails to converge is a measure of the limit point or critical load (see [25]). As explained in [25], when approaching the critical load, the increment in the applied load must be small and the sign of the determinant of the coefficients of the response must be checked. If convergence fails, the load level is over the limit point. But if convergence does not fail and the sign of the determinant changes from what it was at the previous load level, then the load level is also over the limit point. Desired accuracy can be achieved by taking smaller and smaller increments in $\bar{N}_{xx}$. Note that a cost penalty must be paid for improving the accuracy in $\bar{N}_{xx}$ or by this approach. It is also observed that by employing this procedure (algorithm in which $\bar{N}_{xx}$ is known and the response, W_i, f_i, is unknown), no solution can be obtained

past the limit point. Because of this, the new algorithm is employed at this point of the solution procedure. The new algorithm, as already explained, simply changes the role of one of the displacement terms with that of the applied load $\bar{N}_{xx}$. While the first procedure is followed, the most dominant displacement term is identified (or a group of terms). At some level before the limit point, the procedure is switched and a solution is formed that corresponds to an increment in the chosen dominant displacement parameter. To this end, the previous solution is used as an initial solution. The procedure is continued until the entire post-limit point response is obtained. During this phase of the solution procedure, some convergence failures can also occur. These failures can be attributed to one of two reasons: (a) either the increment in the dominant displacement parameter is too large or (b) the NP (number of mesh points) is too small for an accurate description of the response. Both of these can easily be corrected. In this second phase, large increments are purposely used in order to save computer time. If the solution fails to converge, then the increment is automatically reduced.

Numerical integration is used to find the total potential and end shortening. By this solution procedure, the entire load-displacement or load-end shortening curves can be obtained for a given imperfection and each wave number n.

Numerical Results and Discussion

Numerical results are obtained for two geometries, one unstiffened and one stiffened, for axially loaded cylindrical shells.

The geometry for both is described below:

(a) Unstiffened Cylindrical Shell

$R = 4$ in., $t = 0.004$ in., 0.008 in., 0.016 in., 0.050 in.;

$$L = 4 \text{ in.}, 12 \text{ in.}, 20 \text{ in.}, 40 \text{ in.};$$

$$E = 10.5 \times 10^6 \text{psi}; \; \nu = 0.3; \text{ with} \tag{3.59}$$

$$W^o(x,y) = t\xi\left[-\cos\frac{2\pi x}{L} + 0.1 \sin\frac{\pi x}{L}\cos\frac{ny}{R}\right]$$

and SS-3 Boundary Conditions, Eqs. (3.32)

(b) <u>Ring and Stringer-Stiffened Cylindrical Shell</u>

$$R = 4 \text{ in.}; \;\; t = 0.04 \text{ in.}; \;\; L = 4 \text{ in.},$$

$$e_x = \pm 0.24 \text{ in.}; \;\; e_y = \pm 0.12 \text{ in.}; \;\; (+ \text{ for internal stiffeners})$$

$$E = 10.5 \times 10^6 \text{psi}; \;\; \nu = 0.3; \tag{3.60}$$

$$\lambda_{xx} = 0.910; \;\; \lambda_{yy} = 0.455; \;\; \rho_{xx} = 100; \;\; \rho_{yy} = 20; \;\; \text{with}$$

$$W^o_{(x,y)} = h\xi \sin\frac{\pi x}{L}\cos\frac{ny}{R}; \;\; \text{SS-3 Boundary Conditions, Eqs. (3.32)}$$

Before discussing the results, a few more clarifying remarks about the geometry are needed. The unstiffened geometry is taken from [25] and [30]. Note that in these references only the critical load is given and not the complete behavior. This geometry employs, virtually, an axisymmetric imperfection. Note that the non-axisymmetric amplitude is 10% of the axisymmetric amplitude. A smaller value was tried (1% for the non-axisymmetric amplitude) and the response (see Fig. 3.2; R/t = 500) is, for all purposes, identical to that of geometry (a). The only difference is the value for $\bar{N}_{xx_{cr}}$ (limit-point load). This difference only reflects the effect of imperfection amplitude, i.e., for $\xi = 1$, $N_{xx_{cr}} = 12.24$ lbs/in. for 1% non-axisymmetric amplitude, while $N_{xx_{cr}} = 11.44$ lbs/in. for 10% non-axisymmetric amplitude. Note that, in the former case, the maximum imperfection amplitude is 1.01h while in the latter it is 1.10h. The

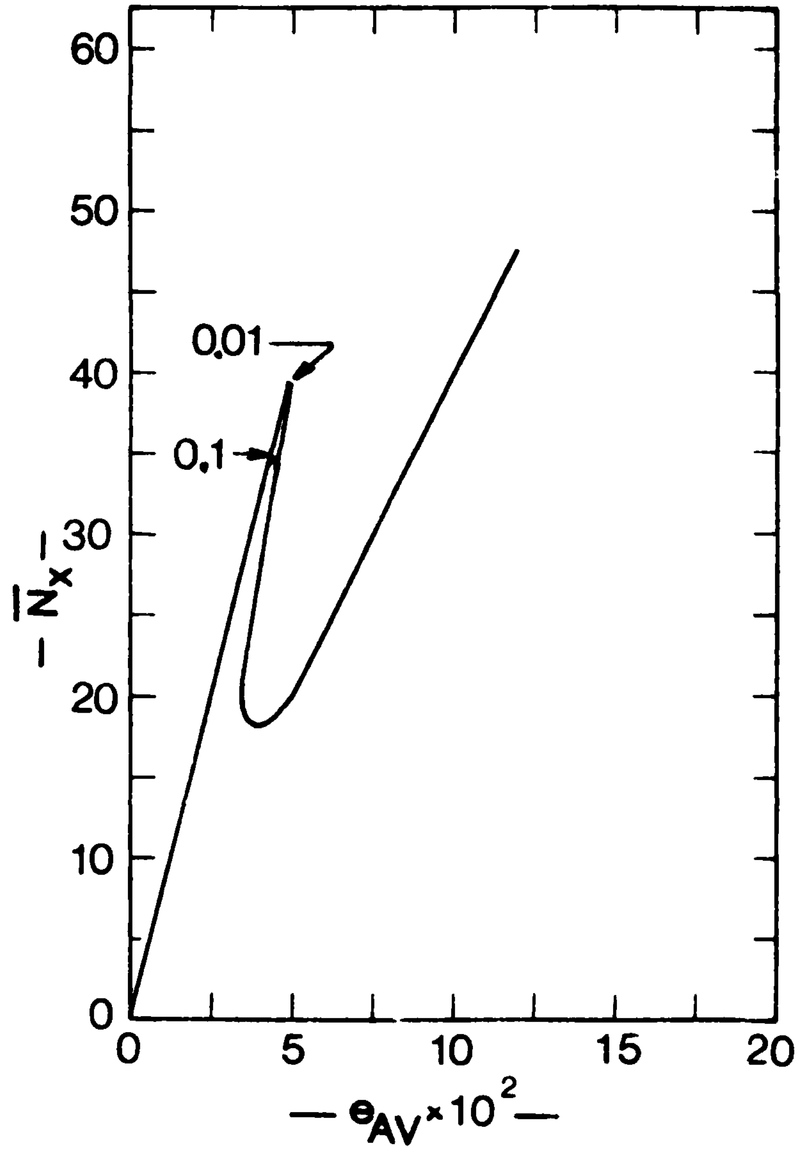

Fig. 3.2 Effect of the Asymmetric Imperfection Amplitude (R/t = 500; L/R = 1; n = 10).

classical load for this case is 25.42 lbs/in. The reason that the 10% amplitude is used in the numerical results obtained is that the higher the non-axisymmetric amplitude, the faster the solution. Moreover, in this geometry, ξ is varied from zero to four, in order to study the effect of imperfection amplitude. Note that for the chosen imperfection,

$$W^{o}_{max}/h = 1.1\xi \tag{3.61}$$

Finally, results are generated for several values of n (number of circumferential full waves). This is needed in order to obtain a clear picture of the complete response.

The stiffened geometry corresponds to examples 14, 16, 18, 19, and 21 of [25]. Again, note that in [25], only limit-point loads were obtained. Moreover, in [25], SS-3 boundary conditions are used, but with $M_{xx} = 0$ [see Eqs. (3.37)]. In the present work, SS-3 with $M_{xx} = a_3\bar{N}_{xx}$ boundary conditions are employed [see Eqs. (3.32)]. In addition to the difference in SS-3 boundary conditions, difference in response for the stiffened geometries lies in the fact that a term $(a_2\bar{N}_{xx})$ is missing from Eq. (20a) of [25]. This omission has been corrected in the present work [see Eq. (3.42)]. The most important results are presented in graphical form and tabular form. In the ensuing discussion, including conclusions, the statements are based on all generated data.

Tables 3.1 and 3.2 present the various unstiffened geometries for which results are obtained (axial compression). Table 3.1 also gives values of critical static and dynamic loads, as well as minimum post-limit point loads and the linear theory (classical) static critical loads. Finally, for each example, it gives the number of mesh points used in the finite difference scheme and the value of n. Table 3.2 summarizes

TABLE 3.1 Axially-Loaded Unstiffened Cylindrical Shells

SS-3; $R = 4.$; $E = 10.5 \times 10^6$; $\nu = 0.3$

$$W^o(x,y) = -\xi t \cos\frac{2\pi x}{L} + 0.1\xi t \sin\frac{\pi x}{L} \cos\frac{ny}{R}$$

No	L in.	t in.	L/R	R/t	ξ	classical N_{xx} lbs/in	N^{ℓ}_{xx} lbs/in	N^{m}_{xx} lbs/in	N^{d}_{xx} lbs/in	n	No. mesh pts.	λ^{ℓ}	λ^{m}	λ^{d}
1	4.0	0.004	1.0	1,000	0.5	25.42	24.44	1.174	20.420	8	55	0.961	0.046	0.803
2							24.13	0.524	17.740	9	55	0.949	0.021	0.698
3							23.46	1.027	14.920	10	55	0.922	0.040	0.587
4							20.71	2.050	13.199	11	35	0.815	0.081	0.519
5							17.55	3.456	11.059	12	35	0.690	0.136	0.435
6							16.61	4.804	10.199	13	35	0.653	0.189	0.401
7							17.17	6.025	10.142	14	35	0.675	0.237	0.399
8	4.0	0.004			1.0		24.21	0.977	19.460	8	55	0.952	0.038	0.766
9							23.44	0.565	16.139	9	55	0.922	0.022	0.635
10							19.22	1.310	12.550	10	35	0.756	0.052	0.494
11							14.16	2.521	9.538	11	35	0.557	0.099	0.375
12							11.86	3.860	8.110	12	35	0.467	0.152	0.319
13							11.34	5.169	7.789	13	35	0.446	0.203	0.306
14	4.0	0.004			4.0		20.98	2.083	15.770	7	35	0.825	0.082	0.620
15							8.47	1.780	6.944	8	35	0.333	0.070	0.273
16							3.30	2.000	2.874	9	35	0.130	0.078	0.113
17							2.10*	---	---	10	35	0.083	---	---
18							2.30*	---	---	11	35	0.091	---	---
19							3.40*	---	---	12	35	0.134	---	---
20							4.50*	---	---	13	35	0.177	---	---
21							5.85*	---	---	14	35	0.230	---	---

TABLE 3.1 **Axially-Loaded Unstiffened Cylindrical Shells** (Cont'd.)

No	L in.	t in.	L/R	R/t	ξ	classical N_{xx} lbs/in	N^{ℓ}_{xx} lbs/in	N^{m}_{xx} lbs/in	N^{d}_{xx} lbs/in	n	No. mesh pts.	λ^{ℓ}	λ^{m}	λ^{d}
22	4.0	0.008	1.0	500	1.0	101.81	98.75	19.01	89.46	6	65	0.970	0.187	0.879
23	↓	↓	↓	↓	↓	↓	91.84	10.90	68.95	7	65	0.902	0.107	0.677
24							65.33	7.80	46.09	8	35	0.642	0.077	0.453
25							43.24	12.29	31.17	9	35	0.425	0.121	0.306
26							35.00	17.98	26.09	10	35	0.344	0.177	0.256
27							35.17	24.10	26.46	11	35	0.345	0.237	0.260
28							39.18	29.97	---	12	35	0.385	0.295	---
29							45.30	34.89	---	13	35	0.445	0.343	---
30	4.0	0.016	1.0	250	1.0	407.23	406.00	---	---	4	35	0.997	---	---
31	↓	↓	↓	↓	↓	↓	362.50	---	291.21	5	35	0.890	---	0.715
32							257.90	42.89	187.42	6	35	0.633	0.105	0.460
33							138.30	51.55	106.41	7	35	0.340	0.127	0.261
34							101.10	72.15	80.61	8	35	0.248	0.177	0.198
35							104.90	97.79	---	9	35	0.258	0.240	---
36							129.00	126.30	---	10	35	0.313	0.310	---
37	4.0	0.05	1.0	80	1.0	3977.00	3500.00	2509.00	2894.00	3	35	0.880	0.631	0.728
38	↓	↓	↓	↓	↓	↓	1356.00	683.80	1103.20	4	35	0.341	0.172	0.277
39							623.80	599.70	---	5	35	0.157	0.151	---
40							675.00*	---	---	6	35	0.169	---	---
41	20.0	0.016	5.0	250	1.0	407.23	621.40	108.40	382.04	3	35	1.526	0.266	0.938
42	↓	↓	↓	↓	↓	↓	292.80	47.79	174.16	4	35	0.719	0.117	0.428
43							353.20	137.00	188.48	5	35	0.867	0.336	0.463
44	32.0		8.0				355.80	132.90	190.00	3	65	0.873	0.326	0.466
45	↓		↓				408.60	178.90	212.12	4	65	1.003	0.439	0.521

TABLE 3.1 **Axially-Loaded Unstiffened Cylindrical Shells** (Cont'd.)

No	L in.	t in.	L/R	R/t	ξ	classical N_{xx} lbs/in	N^{ℓ}_{xx} lbs/in	N^{m}_{xx} lbs/in	N^{d}_{xx} lbs/in	n	No. mesh pts.	λ^{ℓ}	λ^{m}	λ^{d}
46	40.0	0.016	10.0	250	1.0	407.23	910.20	---	545.18	2	65	2.235	---	1.339
47							356.30	159.60	193.21	3	65	0.875	0.392	0.474
48							537.80	175.30	214.39	4	65	1.321	0.430	0.526
49	↓	↓	↓	↓	↓	↓	402.30	---	---	5	65	0.988	---	---
50	4.0	0.05	1.0	80	0.08	3977.00	3287.00	---	---	5	35	0.826	---	---
51		0.008	1.0	500	1.0	101.81	39.38	18.08	27.42	10	35	0.387	0.178	0.269
52	↓	0.05	1.0	80	0.32	3977.00	2234.00	766.70	1595.43	5	35	0.562	0.193	0.401
53	12.0	0.016	3.0	250	1.0	407.23	416.20	46.00	---	4	35	1.020	0.113	---
54							242.40	53.05	162.05	5	35	0.595	0.130	0.398
55							265.00	---	---	6	35	0.651	---	---
56							315.00	---	---	7	35	0.774	---	---
57	↓	↓	↓	↓	↓	↓	361.10	---	---	8	35	0.887	---	---

*Change of slope (not a critical load); see Fig. 3.5.

TABLE 3.2 Summary of Results for Axially-Loaded Unstiffened Shells

R/t	L/R	ξ	Classical $\overline{N}_{xx}$ lbs/in	$\overline{N}^{\ell}_{xx}$ lbs/in	$\overline{N}^{m}_{xx}$ lbs/in	$\overline{N}^{d}_{xx}$ lbs/in	n	λ^{ℓ}	λ^{m}	λ^{d}	n^{ℓ}	n^{m}
1000	1	0.5	25.42	16.61	0.5244	10.1996	8-13	0.653	0.0206	0.401	13	9
1000	1	1.0	25.42	11.34	0.5649	7.7890	8-13	0.446	0.0222	0.306	13	9
1000	1	4	25.42	2.10*	1.7800	2.8740	7-10	0.096	0.0700	0.113	10	8
250	1	1	407.23	101.10	42.89	80.61	6-8	0.248	0.1053	0.198	8	6
250	3	1	407.23	242.40	46.00	162.05	4-5	0.595	0.1130	0.398	5	5
250	5	1	407.23	292.80	47.79	174.16	3-5	0.719	0.1174	0.428	4	4
250	8	1	407.23	355.80	132.90	190.00	3-4	0.873	0.326	0.466	3	3
250	10	1	407.23	356.30	159.60	193.21	3-5	0.875	0.3919	0.474	3	3
1000	1	1	25.42	11.34	0.5649	7.7890	8-13	0.446	0.0222	0.306	13	9
500	1	1	101.81	35.00	7.80	26.09	7-11	0.344	0.0766	0.256	10	8
250	1	1	407.23	101.10	42.890	80.61	6-8	0.248	0.105	0.198	8	6
80	1	1	3977.00	623.08	599.700	---	4-6	0.157	0.151	---	5	5
80	1	0.08	3977.00	3287.00	---	---	5	0.826	---	---	5	5
80	1	0.32	3977.00	2234.00	766.70	1595.43	5	0.562	0.193	0.401	5	5

*change of slope (not a critical load); see Fig. 3.5.

the most important results of the study for axially-loaded unstiffened geometries.

The generated data, appearing in Tables 3.1 and 3.2, are also presented in graphical form and a discussion of the various effects is presented. First the results corresponding to R/t = 1000 are presented and discussed. For this group, L/R is equal to one.

Fig. 3.3 is a plot of $\bar{N}_{xx}$ versus average end shortening for $\xi = 0.5$ (unstiffened geometry). These data are generated for several values of full waves, n, around the circumference. From this figure, it is clear that, as the system is loaded quasi-statically from zero, the load deflection curve is the same and independent of n. The limit-point load, $\bar{N}_{xx_{cr}}$ is definitely n-dependent. It is observed that the value of the total potential corresponding to the lowest limit load and associated n is the smallest of all values corresponding to the same load and different n's (at an equilibrium position). For this value of ξ (which corresponds to $W^o_{max} = 0.55$ h), the limit point occurs at $\bar{N}_{xx} = 16.61$ lbs/in. $[\lambda^{\ell} = (\bar{N}_{xx_{cr}}/\bar{N}_{xx_{c\ell}}) = 0.653]$. In the post-limit point region, the unstable branch shows several changes from n = 13 to n = 12 to n = 11. These changes occur at the unstable portion of the curve. The change from n = 11 to n = 10, etc., to n = 8, occur at the stable portion of the curves. This implies that <u>if one can transverse the post-limit point branches</u>, he would move along the n = 13 (with decreasing load) curve, then along the n = 12 and n = 11 curves (with decreasing load). Then, along the n = 11 curve, the system moves with increasing load until it reaches the n = 10 curve. Then it moves along the n = 10 curve until it intersects the n = 9 curve, etc. In reality, though, under dead weight loading, the system reaches the limit point, and then it snaps-through (violent buckling) towards

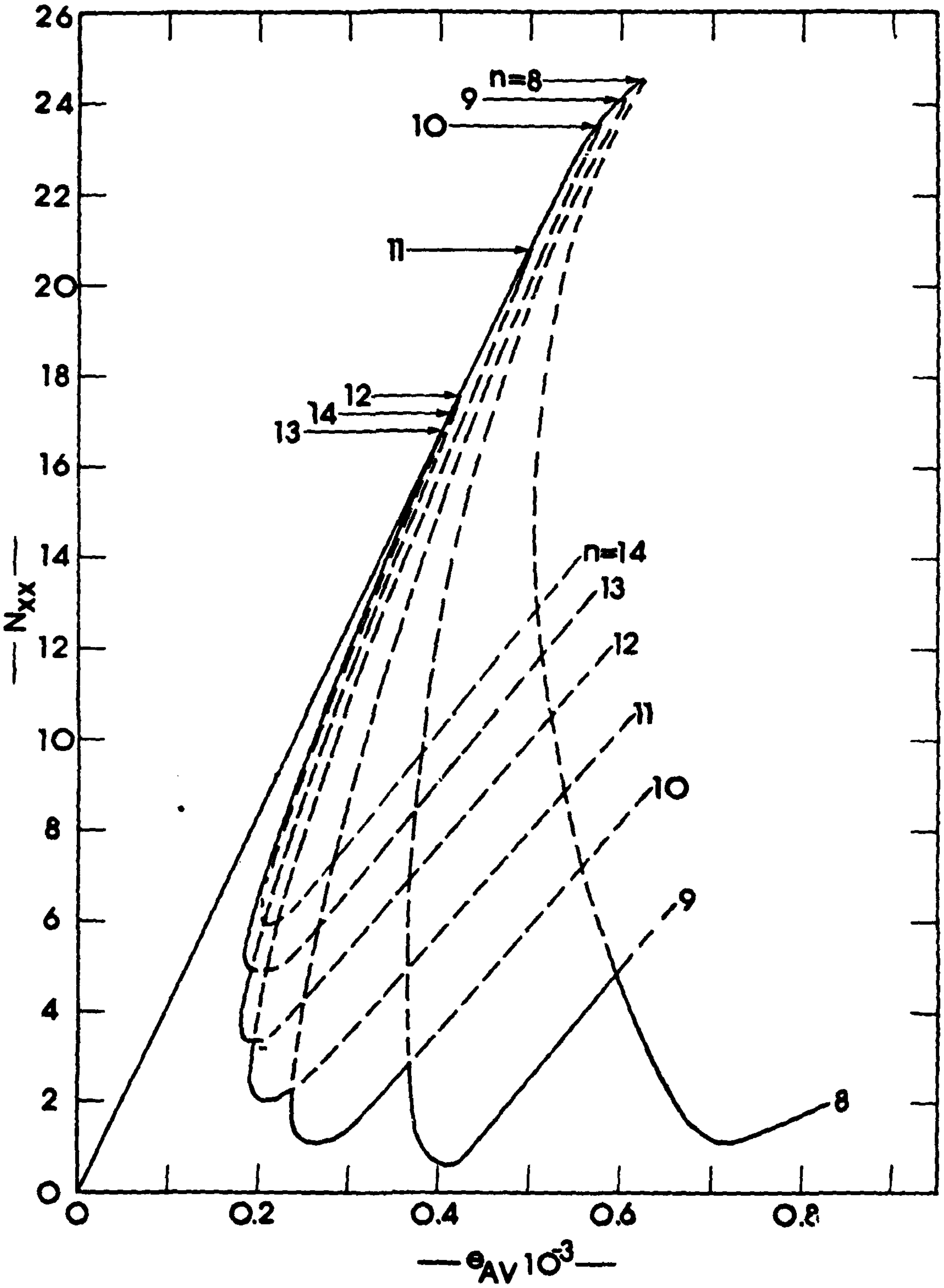

Fig. 3.3 Response of Unstiffened Geometry
(R/t = 1,000; L/R = 1; ξ = 0.5)

far stable equilibrium positions. During the snapping process, it is clear from this figure that the shell experiences changes in the circumferential mode, corresponding to various n-values. This phenomenon has been observed experimentally through high-speed photography for very thin and relatively short cylindrical shells.

Fig. 3.4 presents similar data (as Fig. 3.3), but for $\xi = 1$. The behavior is very similar to that corresponding to $\xi = 0.5$. Note that curves corresponding to $n = 13,...,8$ are shown. Data are generated for $n = 14$ and 15 but are not shown on the figure. No data are generated for $n < 8$, because the minimum load (in the post-limit point region) positions correspond to $n = 9$ for both ξ-values (Figs. 3.3 and 3.4). Clearly, the same observations are made concerning violent buckling with changing circumferential mode. Moreover, data are generated for $\xi = 4$ and plotted on Fig. 3.5. Note that for $n \geq 10$, there is no limit point instability, but for $n = 9$, 8, and 7 there exist limit points. The response, though, as the system is loaded quasi-statically from zero, is along the $n = 10$ path and snapping takes place at the load level corresponding to unstable bifurcation (the $n = 10$ and $n = 9$ paths cross). Even for this imperfection amplitude ($\xi = 4$), violent buckling is predicted with change in circumferential mode. Finally, for the unstiffened geometry, Fig. 3.6 presents the effect of the imperfection amplitude, ξ, on the limit-point load, $\lambda^{\ell} = \bar{N}_{xx_{cr}}/\bar{N}_{xx_{c\ell}}$, and on the minimum load, $\lambda^{m} = \bar{N}_{xx_{min}}/\bar{N}_{xx_{c\ell}}$. It also presents the effect of imperfection amplitude on the dynamic critical load, λ^{d}, for the case of constant load of infinite duration. This effect is discussed in a later section. Note that $\bar{N}_{xx_{min}}$ corresponds to the minimum equilibrium load in the post-limit point region. As it can be seen from Fig. 3.6, the shell is extremely sensitive to initial geometric imperfections

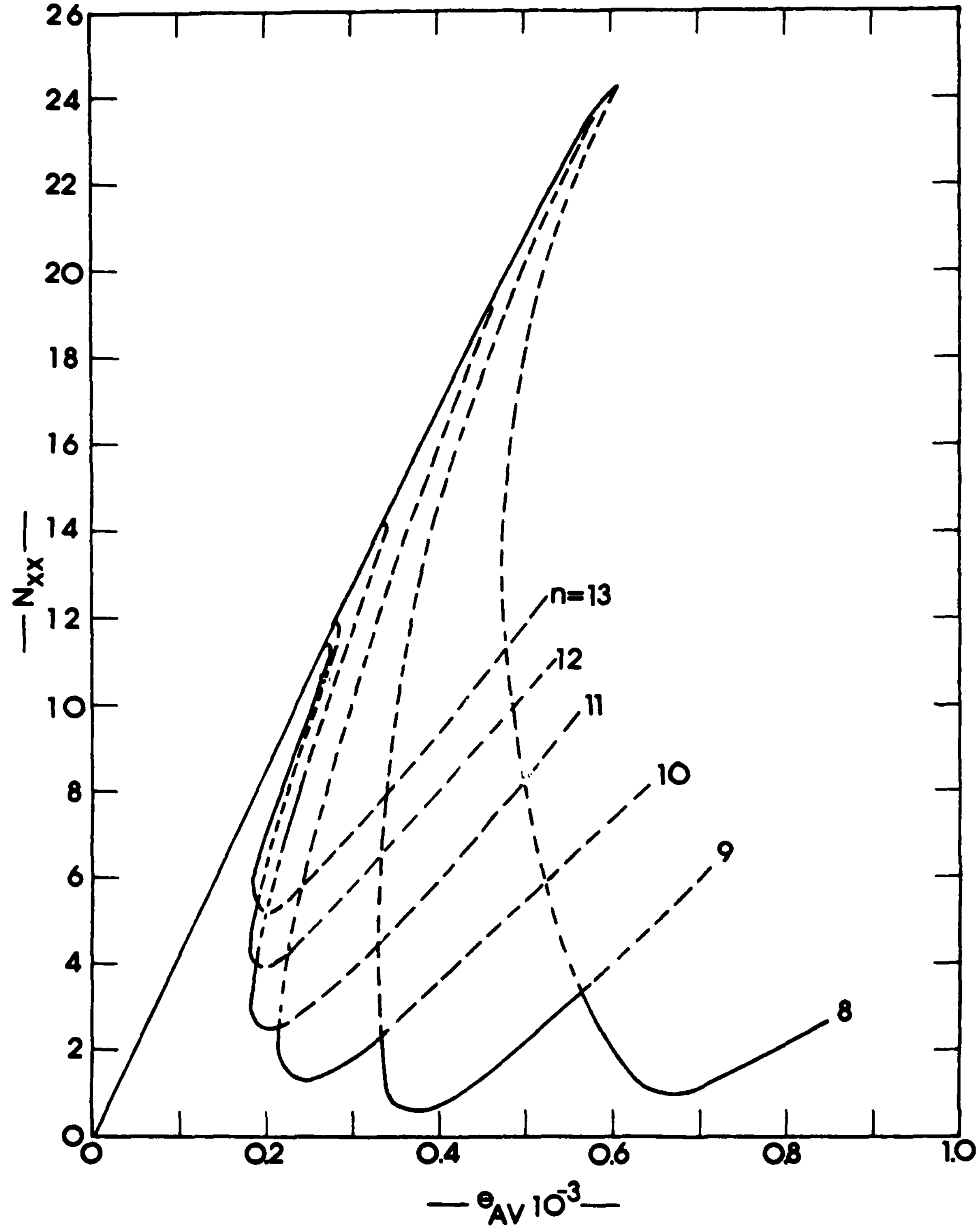

Fig. 3.4 Response of Unstiffened Geometry
(R/t = 1,000; L/R = 1; ξ = 1.0)

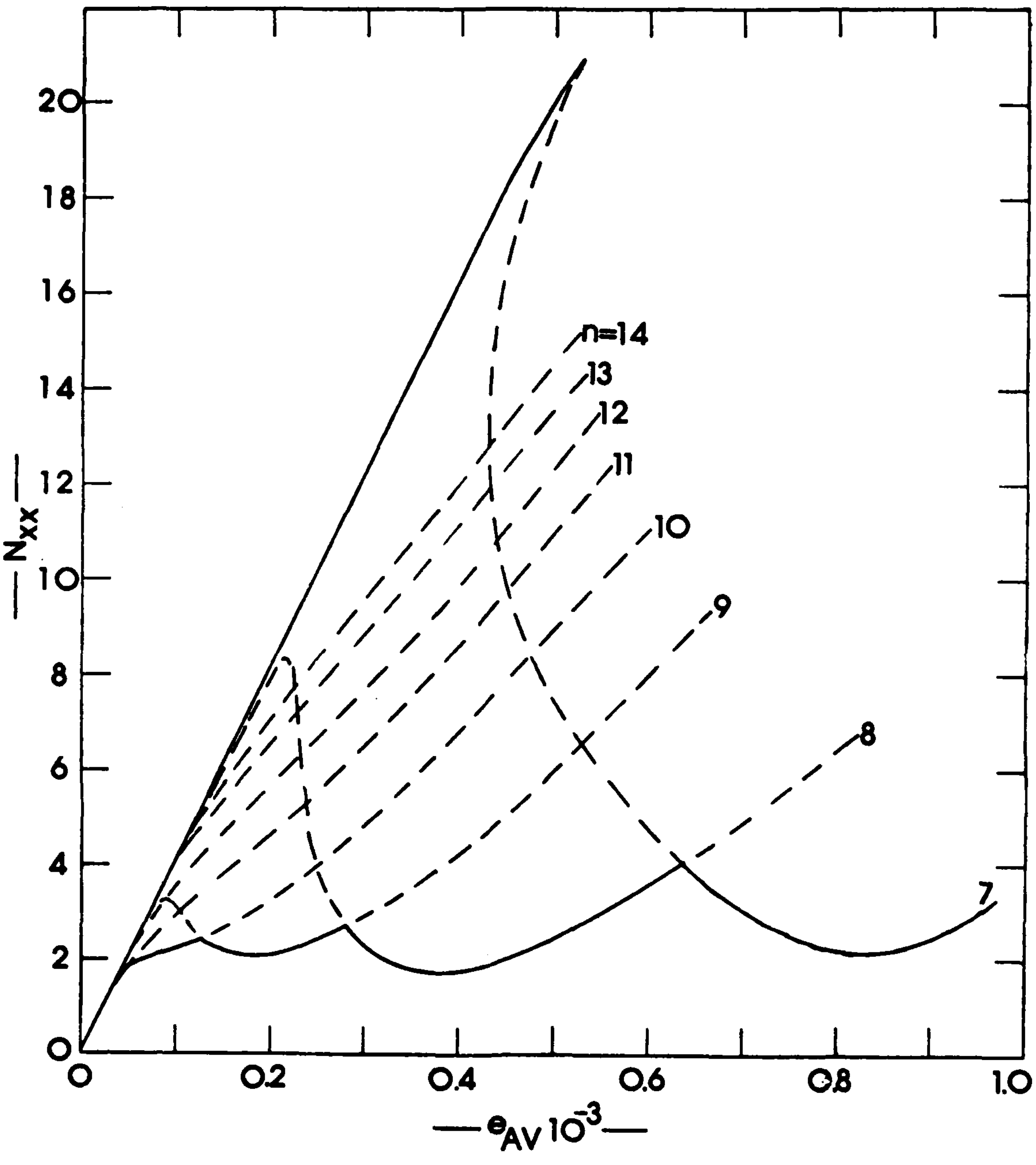

Fig. 3.5 Response of Unstiffened Geometry (R/t = 1,000; L/R = 1; ξ = 4.0)

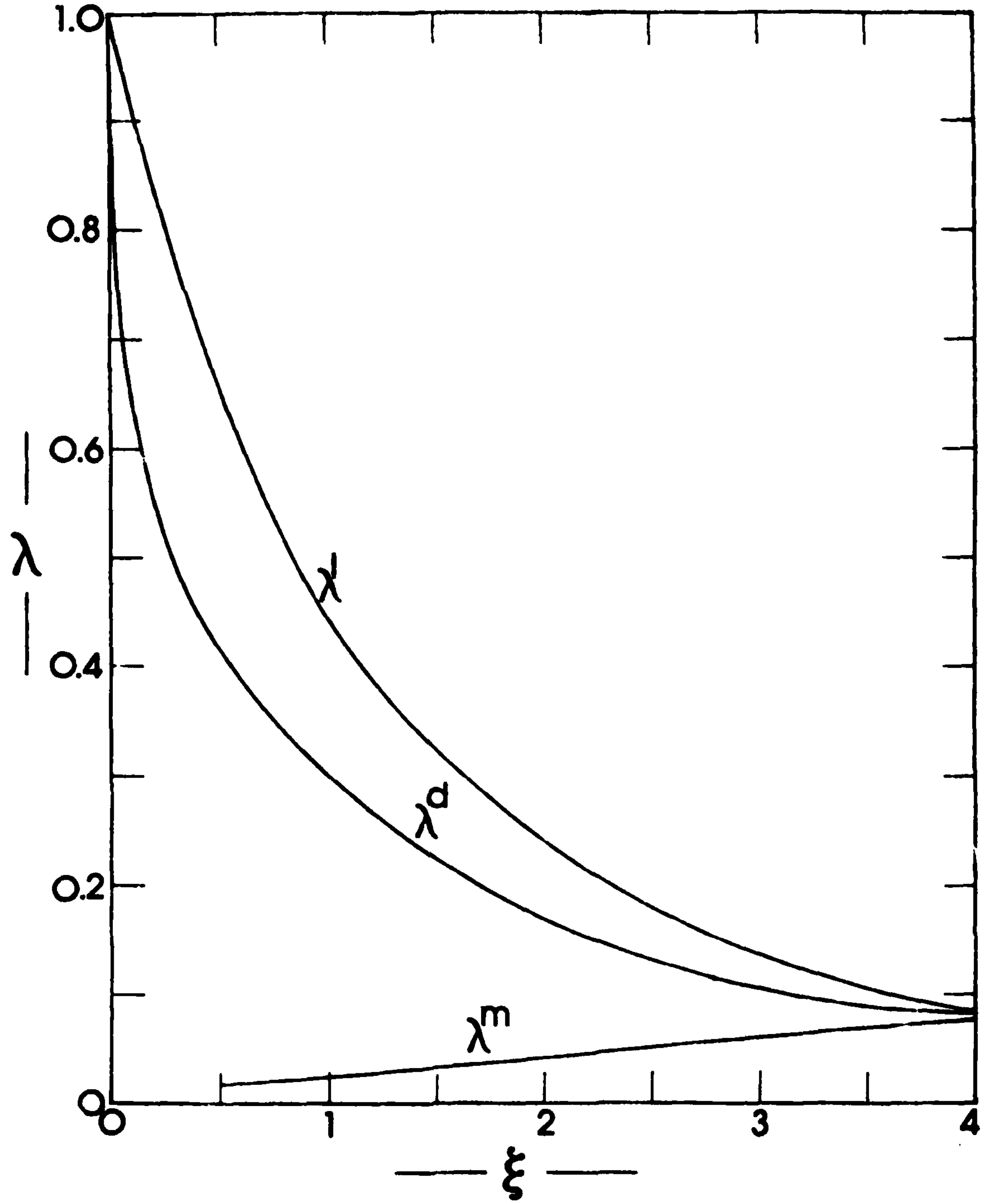

Fig. 3.6 Plots of Load Parameters (λ^{ℓ}: limit point; λ^{d}: critical dynamic; λ^{m}: minimum post-limit point) versus Imperfection Amplitude Parameter.

(of virtually axisymmetric shape). Note that at $\xi = 0.84$, λ^{ℓ} is equal to one half. Since $W^{o}_{max} = 1.1\xi\ h$, then $\lambda^{\ell} = 0.5$, when $W^{o}_{max} = 0.924\ h$. At $\xi = 3.5$, $\lambda^{\ell} = 0.1$, and at $\xi = 4$ the values of λ^{ℓ} and λ^{m} are almost the same. This means that for $\xi \geq 4$, there is no possibility of snap-through buckling. The cylindrical shell simply deforms, with bending, from the initial application of the load. Finally, Fig. 3.7 presents a composite of Figs. 3.3-3.5, and it includes pre-limit and post-limit point behavior for $\xi = 0.5$, 1.0 and 4.0.

Fig. 3.8 is similar to Fig. 3.4 but for R/t = 500. Moreover, Figs. 3.9 and 3.10 fall in the same catagory. These geometries correspond to Examples 22-40, which along with Examples 8-13 serve to study the effect of R/t on the shell response characteristics. Note that for all of these examples, $\xi = 1.0$ and L/R = 1.

Clearly, from Fig. 3.8, it is seen that the response characteristics of the shell are very similar to those corresponding to R/t = 1000 (Fig. 3.4). The only difference is that the wave number n corresponding to both the limit point (n = 11) and the minimum post-limit point equilibrium load (n = 8) are smaller than the ones for R/t = 1000. According to Fig. 3.4 these wave numbers are n = 13 and n = 9 respectively. Note from Figs. 3.9 and 3.10 that this trend continues as R/t decreases, and for R/t = 80 n = 5 corresponds to both loads. The composite response is shown on Fig. 3.11.

Next, the effect of L/R is examined through examples 30-36, 41-49, and 53-57 (see Table 3.1). All of these geometries correspond to $\xi = 1$ and R/t = 250, and L/R varies from one to ten. The results of this study are presented graphically on Figs. 3.9, 3.12, 3.13 and in the composite of Fig. 3.14. It is seen from these figures that as L/R increases, the

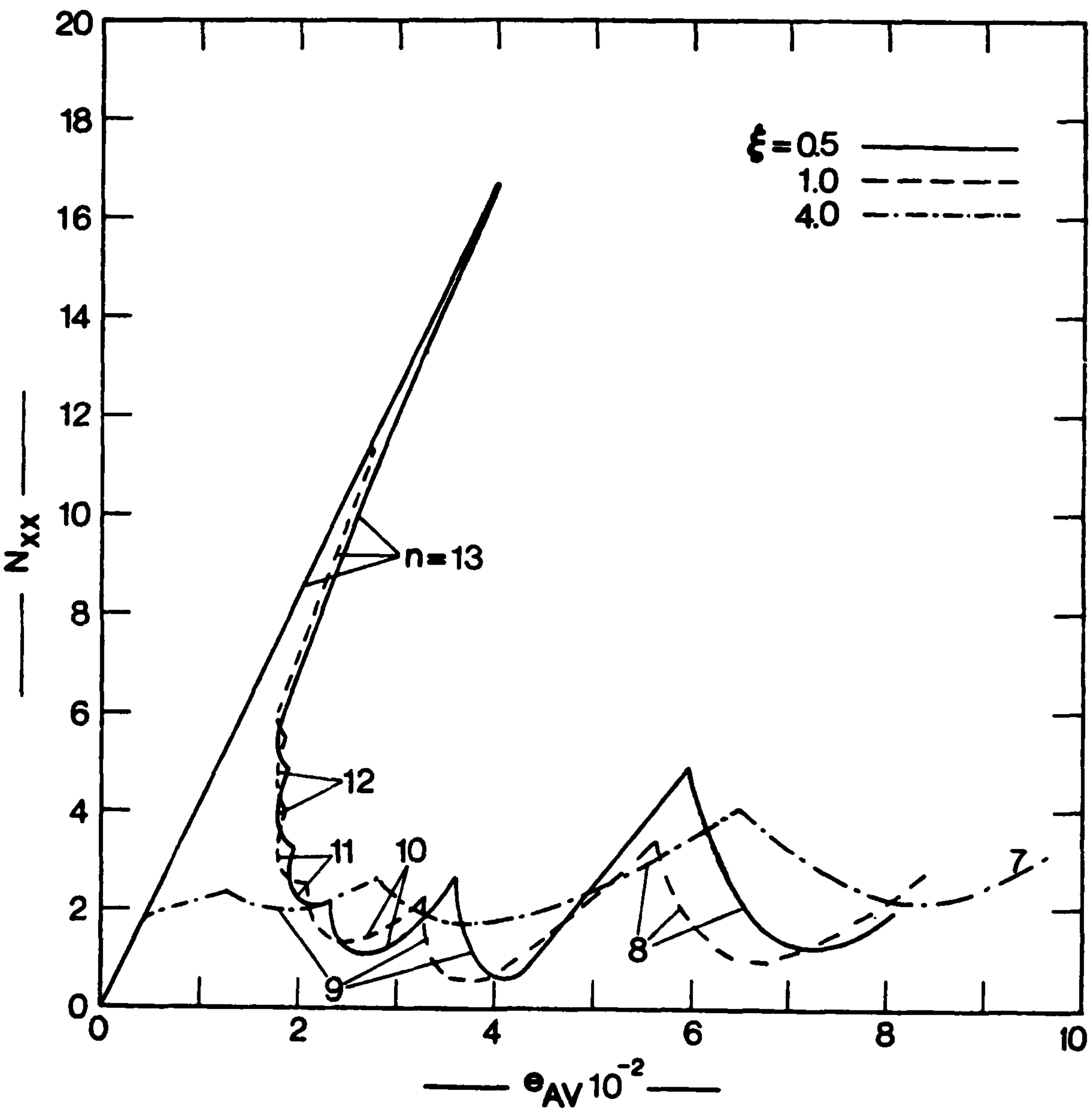

Fig. 3.7 Pre- and Post-Limit Point Response of Unstiffened Geometry (R/t = 1,000; L/R = 1)

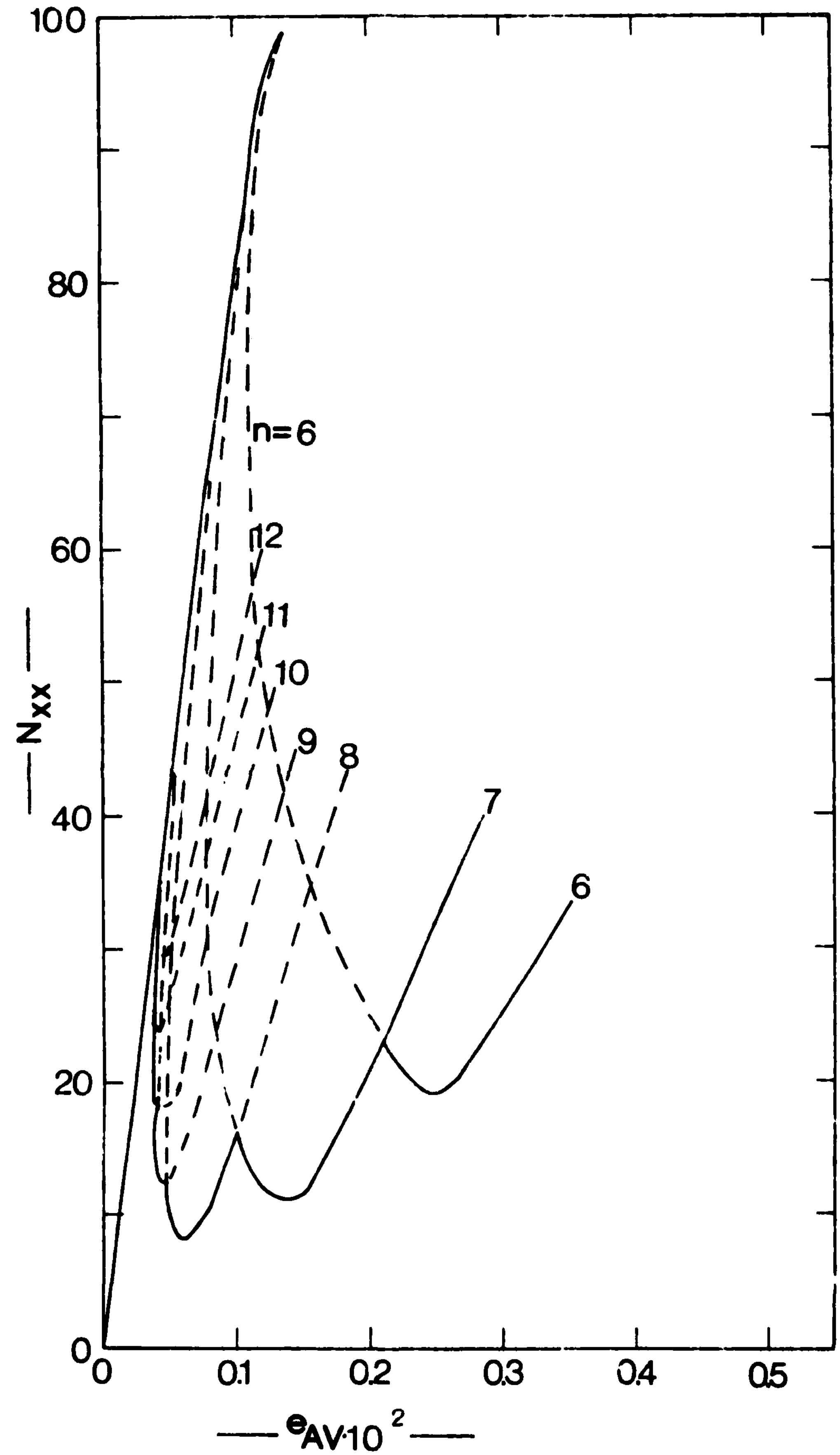

Fig. 3.8 Response of Unstiffened Geometry (R/t - 500; L/R = 1, ξ = 1.0)

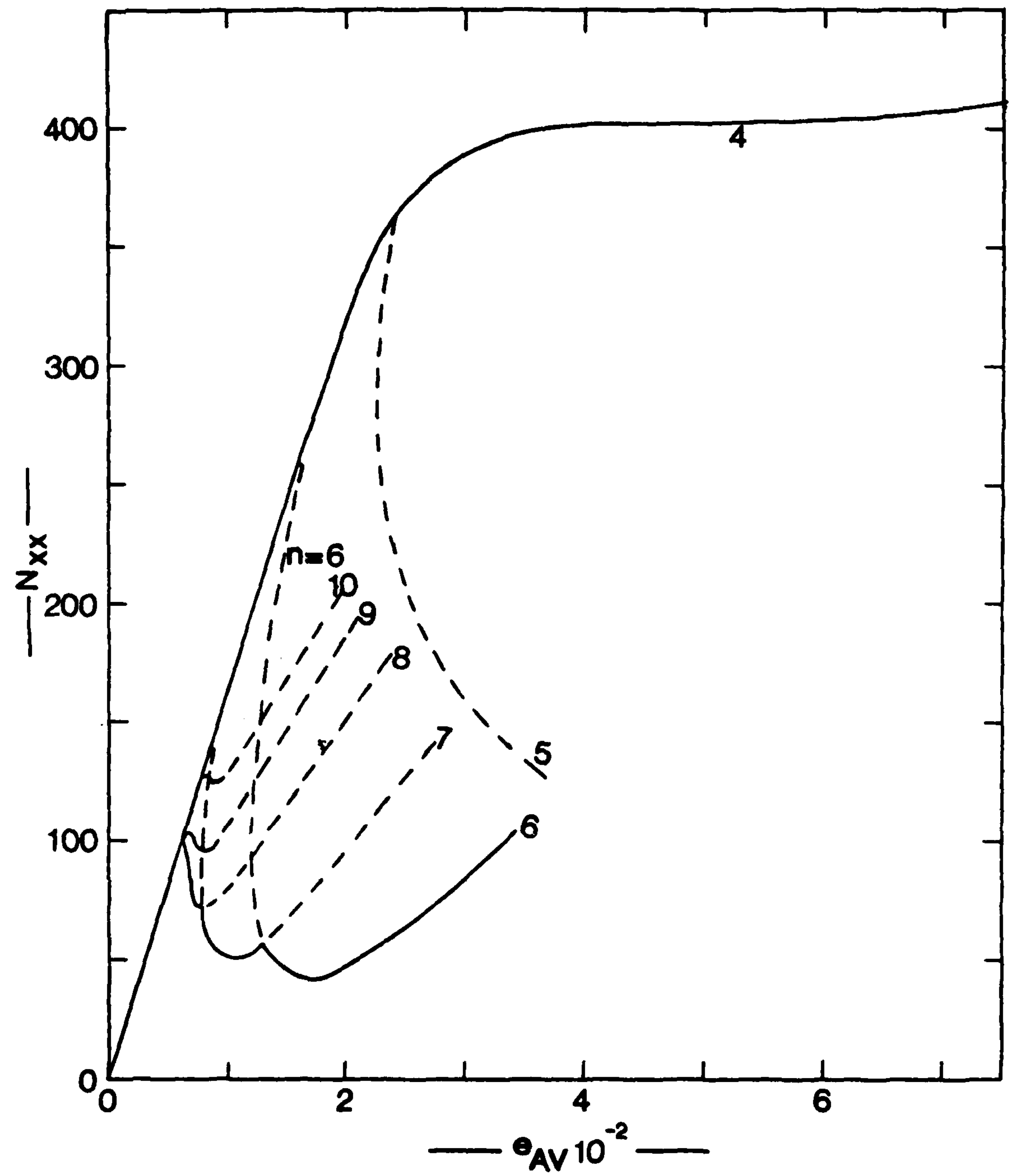

Fig. 3.9 Response of Unstiffened Geometry
(R/t = 250; L/R = 1; ξ = 1.0)

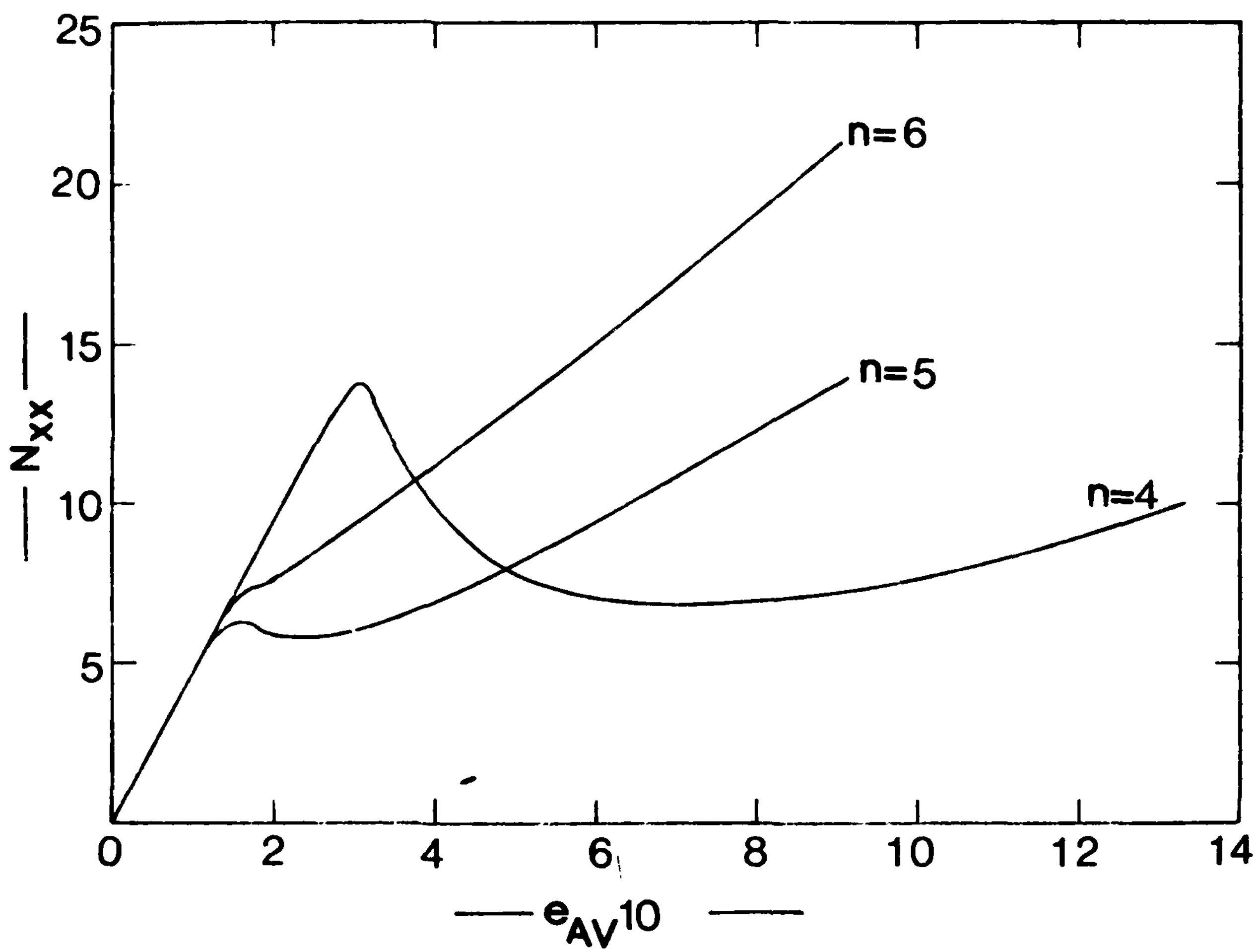

Fig. 3.10 Response of Unstiffened Geometry
(R/t = 80; L/R = 1; ξ = 1.0)

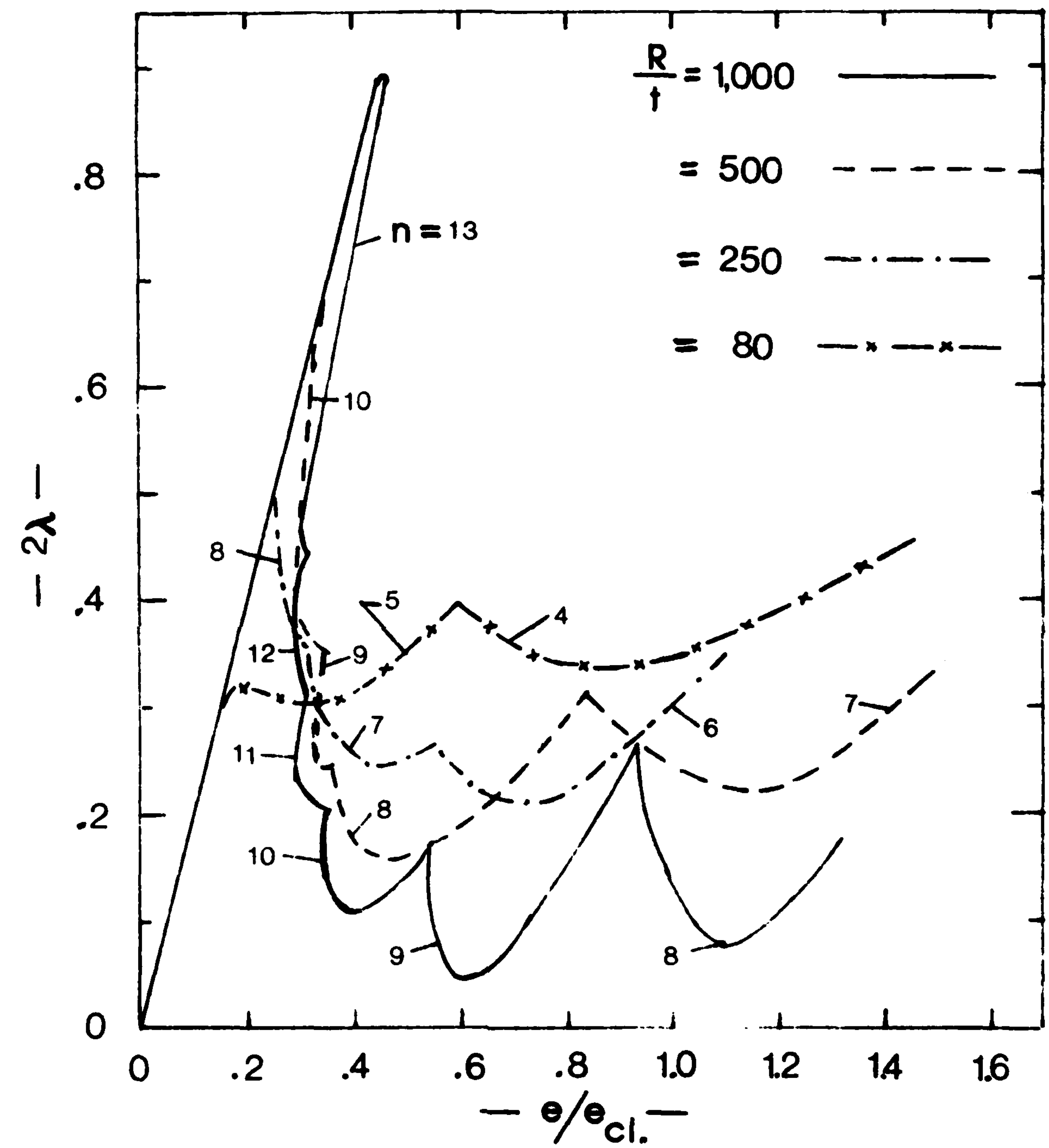

Fig. 3.11 Effect of R/t on the Response of Unstiffened Geometry (L/R = 1; ξ = 1.0)

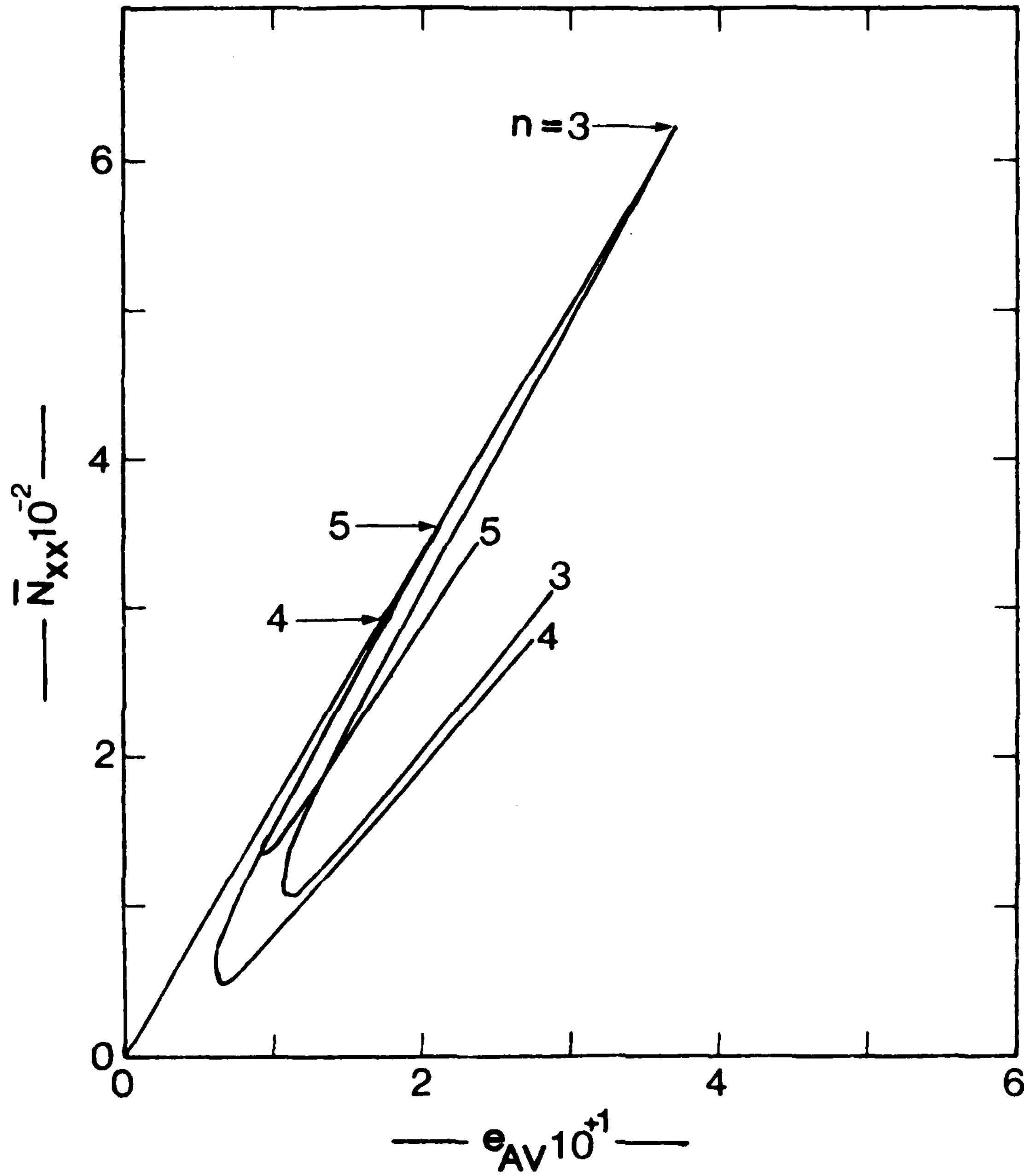

Fig. 3.12 Response of Unstiffened Geometry
(R/t = 250; L/R = 5; ξ = 1.0)

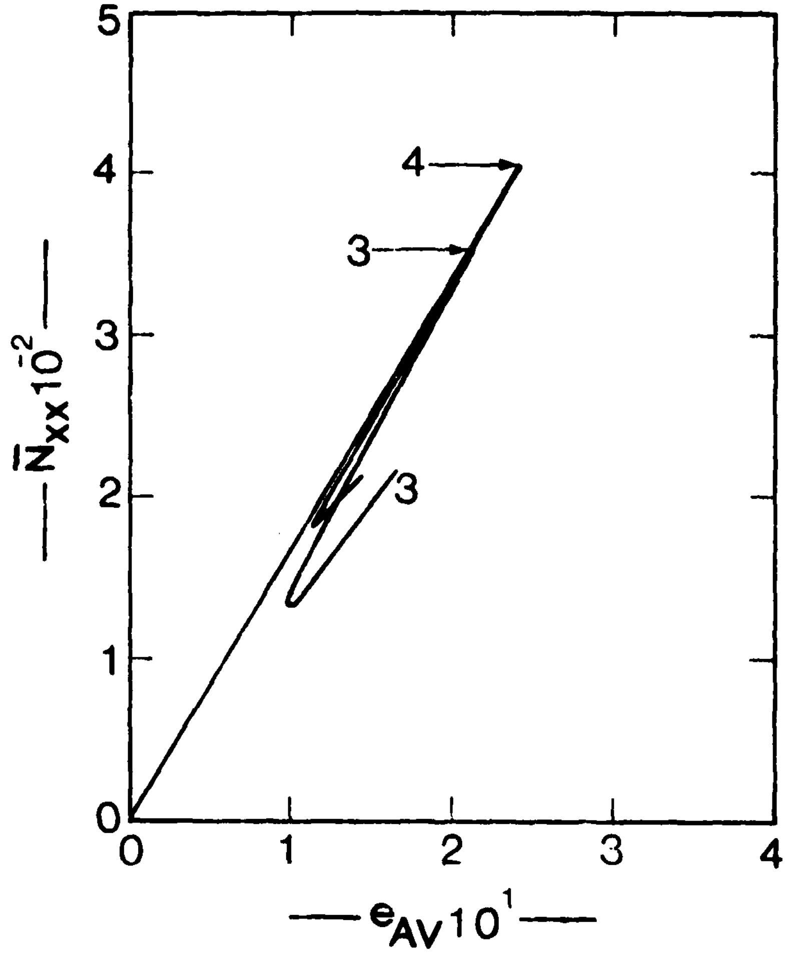

Fig. 3.13 Response of Unstiffened Geometry
(R/t = 250; L/R = 8; ξ = 1.0)

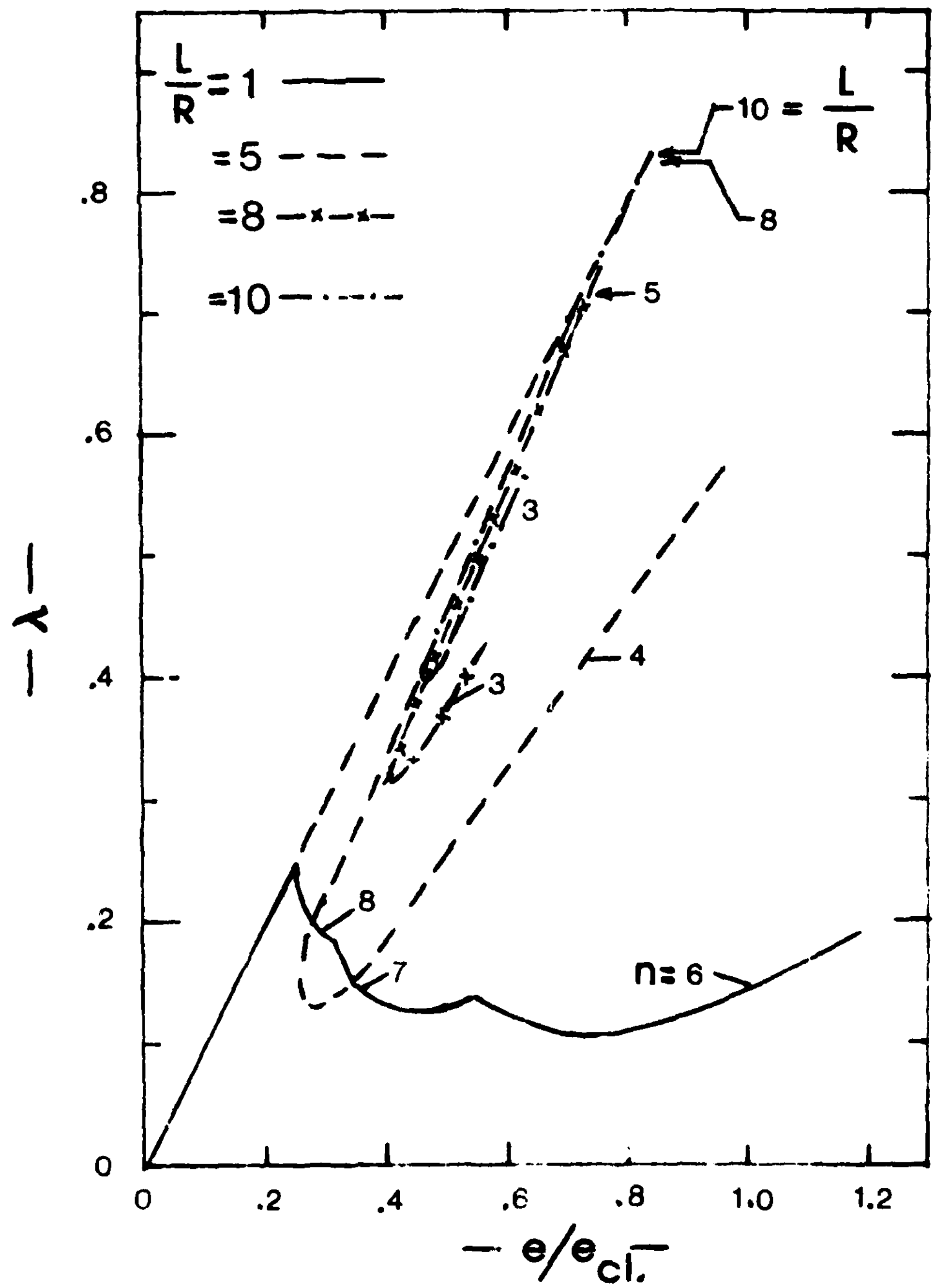

Fig. 3.14 Effect of L/R on the Response of Unstiffened Geometry (R/t = 250; ξ = 1.0)

entire static equilibrium response corresponds to one wave number ($n = 4$ for $L/R = 5$, $n = 3$ for $L/R = 8$ and 10). Note also that the n-value decreases as L/R increases. This result seems reasonable. In addition, the sensitivity of the shell decreases as L/R increases. This latter effect is better shown on Fig. 3.15. The dynamic results are discussed in a later section.

Finally, Fig. 3.16 presents the effect of R/t on the limit point load, minimum post-limit point load and dynamic critical load. There are two sets of curves, one solid and one dashed. The solid curves correspond to $\xi = 1$ and they imply change in the imperfection amplitude as R/t changes, since the data are generated for a constant R value (4 in.). The dashed line set corresponds to the same imperfection amplitude regardless of the value of the thickness. Note that, when $R/t = 1000$, $t = .004$ in. since $R = 4$ in. From the amplitude of the imperfection, one may relate the solid curve to $\xi = 1$ and the dashed curve to $\xi = 0.016/0.004 = 4$. Thus, it is very reasonable that λ^{ℓ} corresponding to $\xi = 4$ is much smaller than λ^{ℓ} corresponding to $\xi = 1$. On the other end of the curve, say $R/t = 100$, the opposite is true. For this value of R/t, $t = 0.04$ in. Then the dashed line curve corresponds to $\xi = \frac{.016}{.040}$, or $\xi = 0.4$, and λ^{ℓ} corresponding to $\xi = 0.4$ is expected and is larger than λ^{ℓ} corresponding to $\xi = 1.0$ (solid curve).

For the stiffened geometries, the results are presented on Figs. 3.17-3.19.

The classical values for $\bar{N}_{xx}$ are 35,220 lbs/in. for external positioning of the stiffeners and 19,790 lbs/in. for internal. The geometric imperfection for the stiffened geometries is not axisymmetric but symmetric with respect to y [see Eqs. (3.64)]. This shape is similar to the classical buckling mode, provided that $n = 4$.

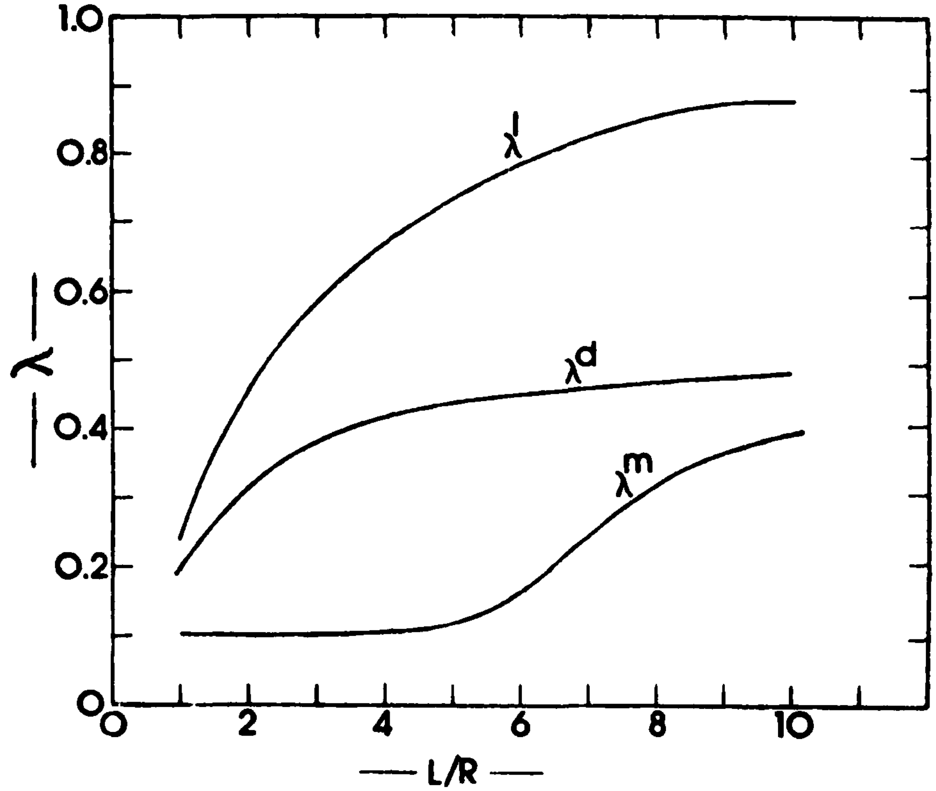

Fig. 3.15 Plots of Load Parameters (λ^{ℓ}; λ^d; λ^m) versus L/R (R/t = 250; ξ = 1)

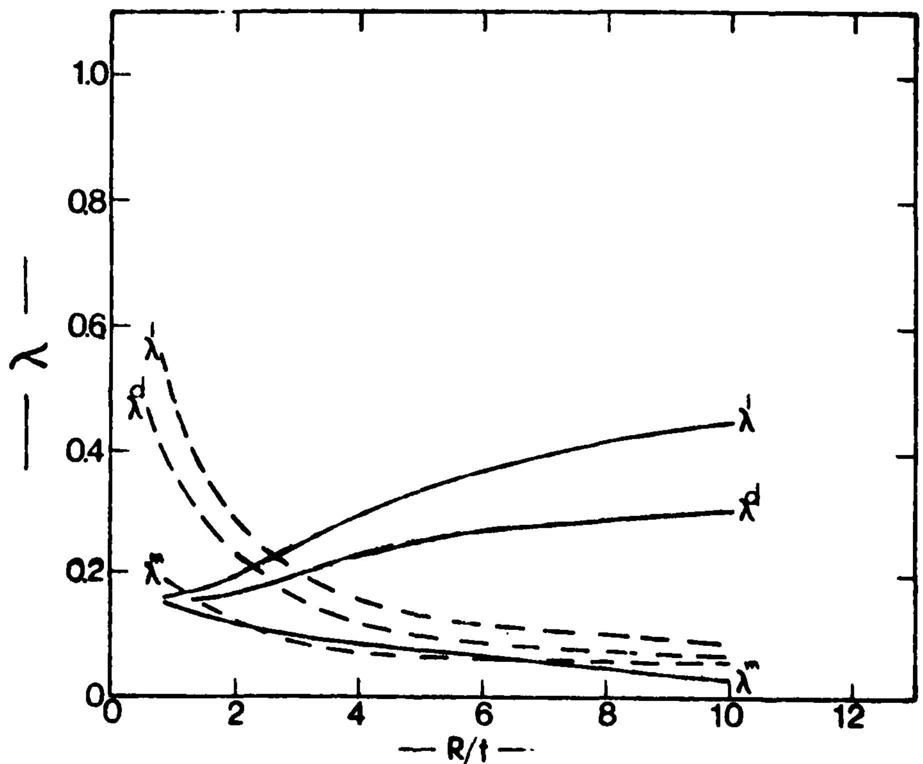

Fig. 3.16 Plots of Load Parameters (λ^{ℓ}; λ^d; λ^m) versus R/t (L/R = 1; solid curve ξ = 1; dashed line constant imperfection amplitude, ξ = 0.016/t).

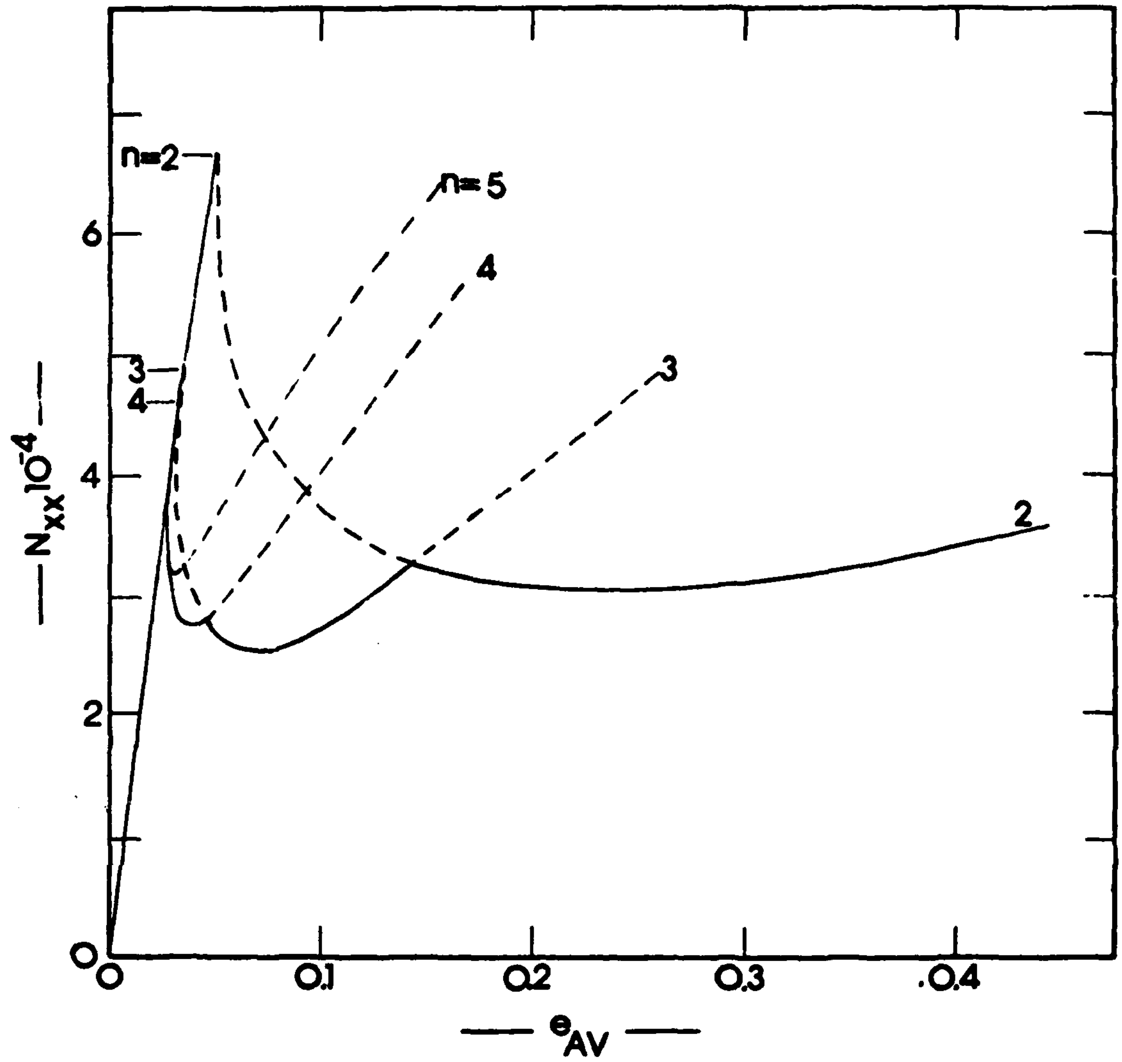

Fig. 3.17 Response of Externally Stiffened Geometry (ξ = 1; Axial Load)

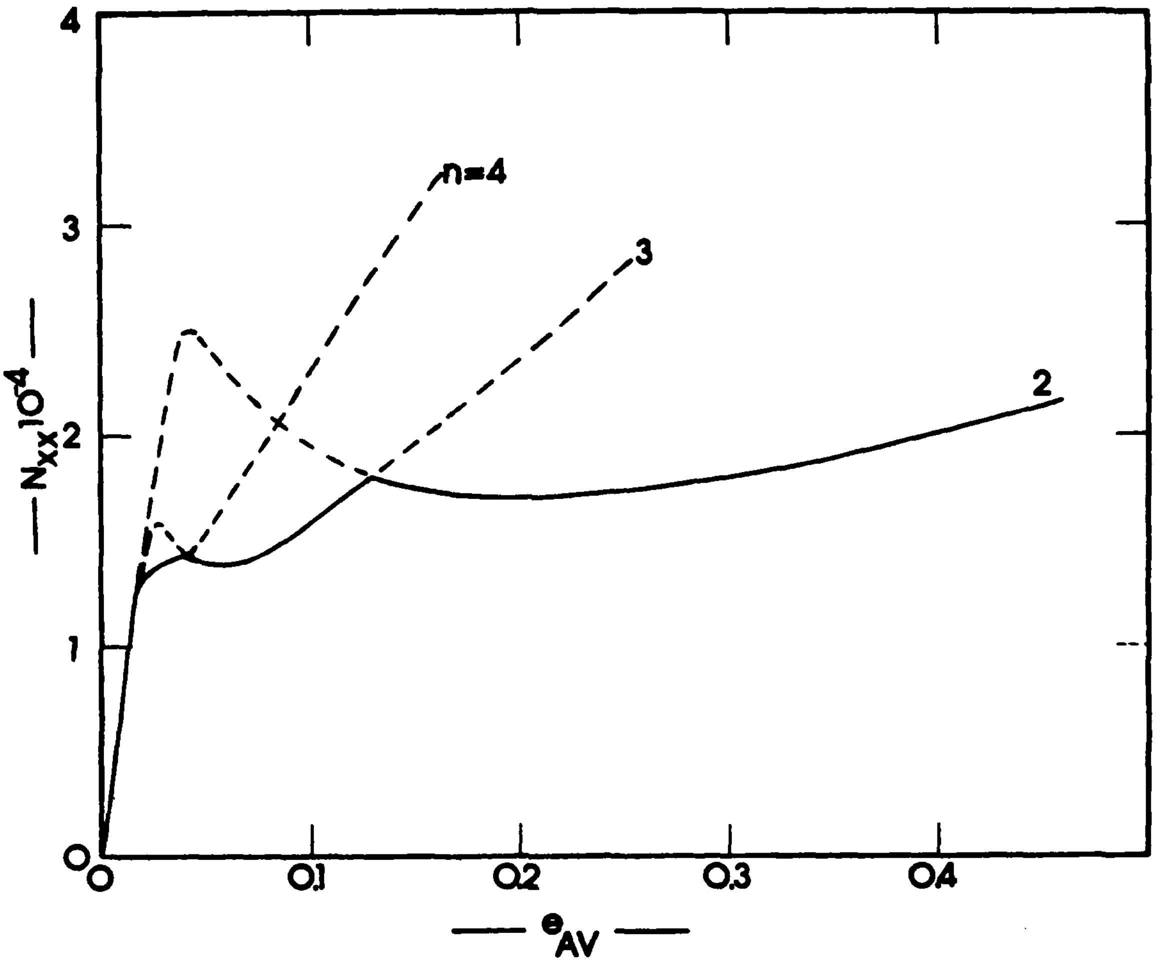

Fig. 3.18 Response of Externally Stiffened Geometry (ξ = 4; Axial Load)

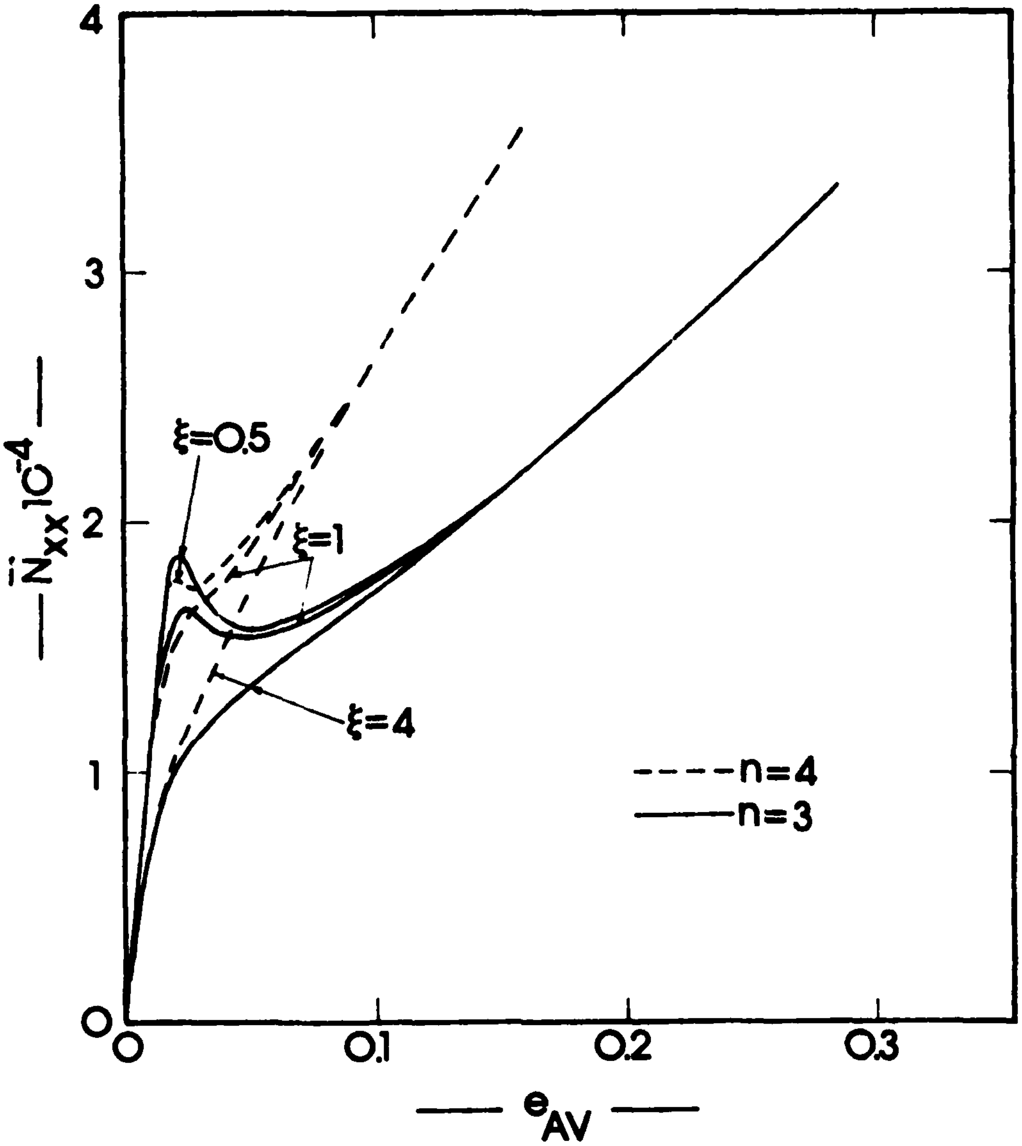

Fig. 3.19 Response of Internally Stiffened Geometry (Axial Load)

The results for the external positioning of the stiffeners are presented on Figs. 3.17 and 3.18 for ξ equal to one and four respectively. It is seen from these two figures that the response is similar to the unstiffened geometry (Figs. 3.3-3.5), but the number of full circumferential waves is smaller (this is an effectively much thicker thin shell). Note that the lowest limit point corresponds to $n = 4$ for $\xi = 1$. On the other hand, for $\xi = 4$, the mode changes from $n = 4$ to $n = 3$ and snapping occurs, because of the existence of an unstable bifurcational branch. In this case also, a change in mode is observed during snap-through buckling. Another important similarity to the unstiffened shell behavior is that this configuration is sensitive to initial geometric imperfections. Note that when $\xi = 1$ (which means that $W^o_{max} = h$), $\bar{N}_{xx_{cr}} = 26,200$ lbs/in. or $\lambda^{\ell} = 0.46$. The externally stiffened shell is not as sensitive as the unstiffened thinner shell, but it is sensitive to initial geometric imperfections.

The results for the internally stiffened configuration are shown in Fig. 3.19. The dashed lines correspond to $n = 4$ and the solid lines to $n = 3$. Data for other n-values need not be shown on this figure. The three sets of curves correspond to $\xi = 0.5$, 1, and 4. Note that, for $\xi = 0.5$, limit point instability occurs at $\bar{N}_{xx} = 17,800$ lbs/in. with $n = 4$. Also note that during snap-through buckling, a change of circumferential mode occurs (to $n = 3$). The minimum equilibrium load in the post-limit point region corresponds to $n = 3$. On the other hand for $\xi = 1$, snap-through buckling occurs at $\bar{N}_{xx} = 16,400$ lbs/in. because of the existence of an unstable bifurcated branch (corresponding to $n = 3$). The minimum equilibrium load for $\xi = 1$ also corresponds to $n = 3$. Finally, there is no possibility of a snapping phenomenon for $\xi = 4$, neither through the

existence of a limit point nor through the existence of an unstable bifurcated branch. It is observed that this configuration is not very sensitive to initial geometric imperfections. For $\xi = 0.5$, $\lambda^{\ell} = 0.9$ and for $\xi = 1.0$, $\lambda^{\ell} = 0.84$. This is attributed to two reasons: (a) internally stiffened configurations are less sensitive than externally stiffened ones and stiffened configurations are less sensitive than unstiffened ones, and (b) for this reported case, SS-3 with $M_{xx} = a_3\bar{N}_{xx}$ boundary conditions are used, which has a stabilizing effect. The primary reason, though, is the former.

Numerical results are also obtained for a ring-stiffened geometry under pressure. This is the same as Example 1 of Ref. 31.

(c) <u>Ring-Stiffened Cylindrical Shell</u>

$$\begin{gathered} L = R = 4 \text{ in.}; \quad t = 0.04 \text{ in.}; \quad E = 10.5 \times 10^6 \text{psi} \\ \nu = 0.3; \quad e_x = \lambda_{xx} = 0; \quad \lambda_{yy} = 0.91; \quad \rho_{yy} = 100; \\ e_y = 0.24 \text{ in.}; \quad \text{classical } p_{cr} = 4827 \text{ psi} \\ w^o(x,y) = \xi t\left(\sin\frac{\pi x}{L} + 0.1 \sin\frac{\pi x}{L}\cos\frac{ny}{R}\right); \\ \xi = 0.1,\ 1.0 \text{ and } 4.0. \end{gathered} \tag{3.62}$$

The results of this study are presented in graphical form on Figs. 3.20-3.22.

Fig. 3.20 shows a plot of the pressure, p, versus the average end shortening for $\xi = 1$ and $n = 3$. There are three plots shown on this figure which serve to check the effect of K [see Eq. (3.40)] and the number of mesh points, N.P., on the convergence of the solution. This effect, as characterized by Fig. 3.20, is typical for all ξ and n values.

Clearly, neither effect is significant for pre-limit point behavior and post-limit point behavior up to the minimum load. Beyond this range, the effect of NP is very small, while the effect of K can be significant. Therefore, if one is interested in the response characteristic up to the minimum post-limit point load, both effects are insignificant. In this particular study, one is interested in establishing limit-point loads (critical static loads) and dynamic critical loads which depend on accurately predicting the response in the unstable portion of the post-limit point behavior. (The value of the modified total potential goes to zero in this range if a critical dynamic load exists.) The conclusion is that K = 1 and NP = 35 suffice for this study, since the cpu time increases rapidly with increases in both K and NP.

Fig. 3.21 shows the effect of n for $\xi = 1$. This effect is the same for the other ξ values, and n = 3 characterizes the true response of the ring-stiffened shell. Fig. 3.22 shows the response of the shell for all three values of ξ (and n = 3). Note that for $\xi = 0.1$ and 1.0, the shell expands in the axial direction (negative end shortening) up to the limit point and then it starts to contract. For $\xi = 4$, initially there is an expansion, but contraction commences before reaching the limit point. Note also that this configuration is rather sensitive to initial geometric imperfection (for $\xi = 0.1$, $\lambda^{\ell} = .94$; for $\xi = 1.0$, $\lambda^{\ell} = 0.80$; for $\xi = 4.0$, $\lambda^{\ell} = 0.59$). Note also that the agreement between the value reported in Ref. 31 for λ^{ℓ} and $\xi = 1$ and the present one is very good. Moreover, the value of n (=3) is the same for both.

Critical Conditions for Sudden Application of the Loads.

As stated previously, a critical dynamic condition exists if the modified total potential, $U_{T_{mod}}$, becomes zero at an unstable static equili-

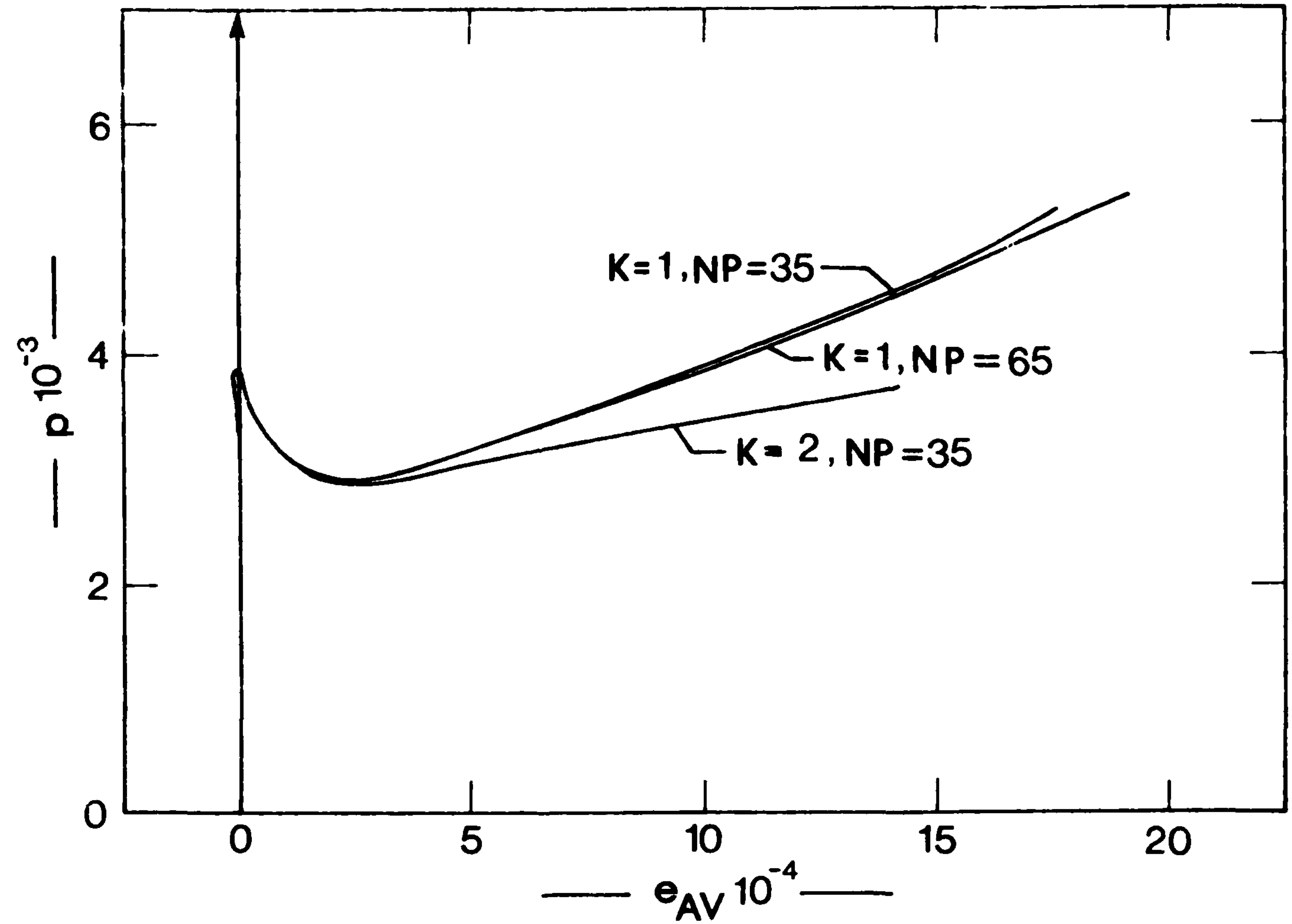

Fig. 3.20 Effect of K and NP (Fourier terms and Mesh Points) on the Response of the Ring-Stiffened Geometry ($n = 3$; $\xi = 1$; pressure load)

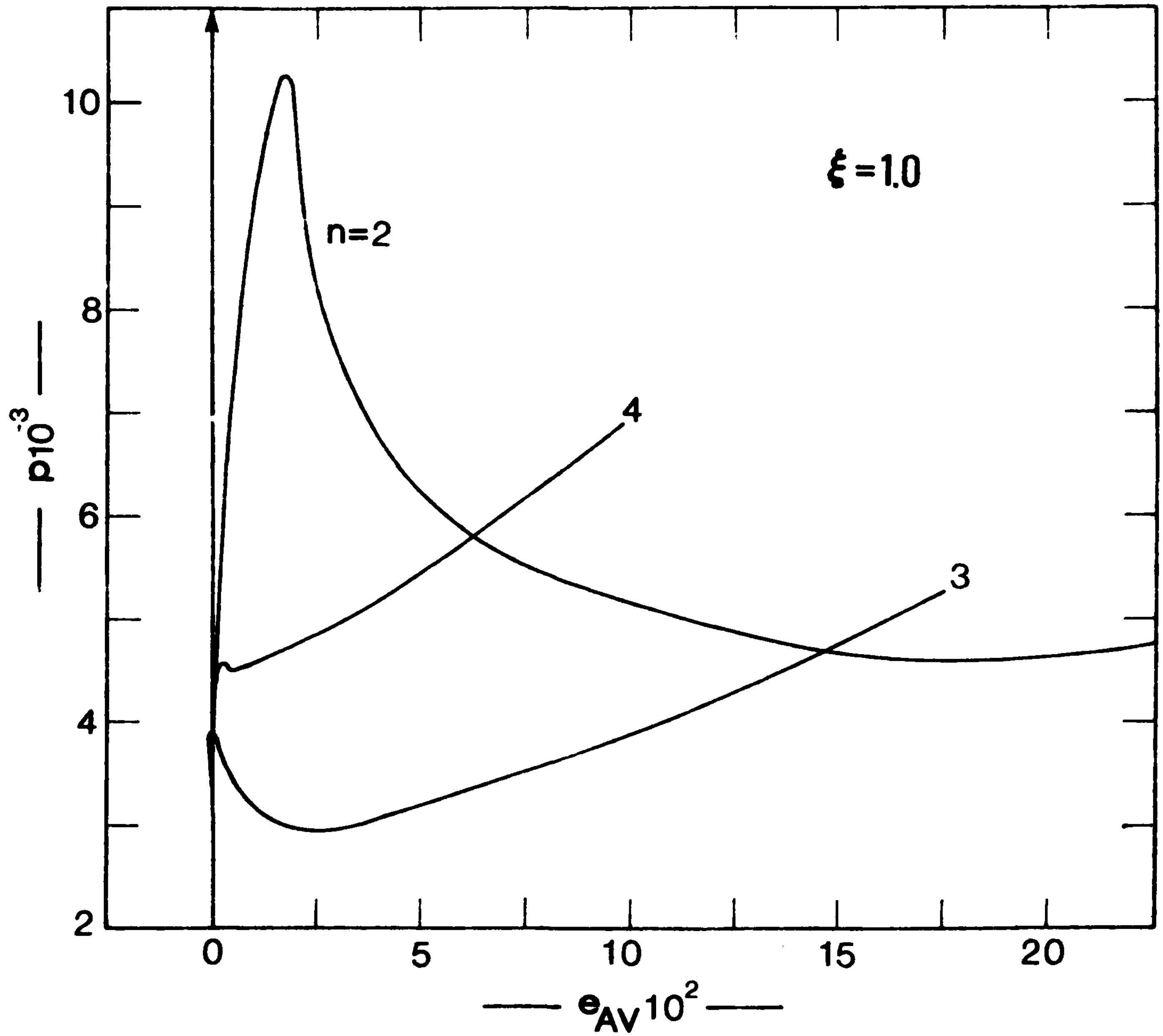

Fig. 3.21 Response of Ring-Stiffened Geometry (pressure load; $\xi = 1$)

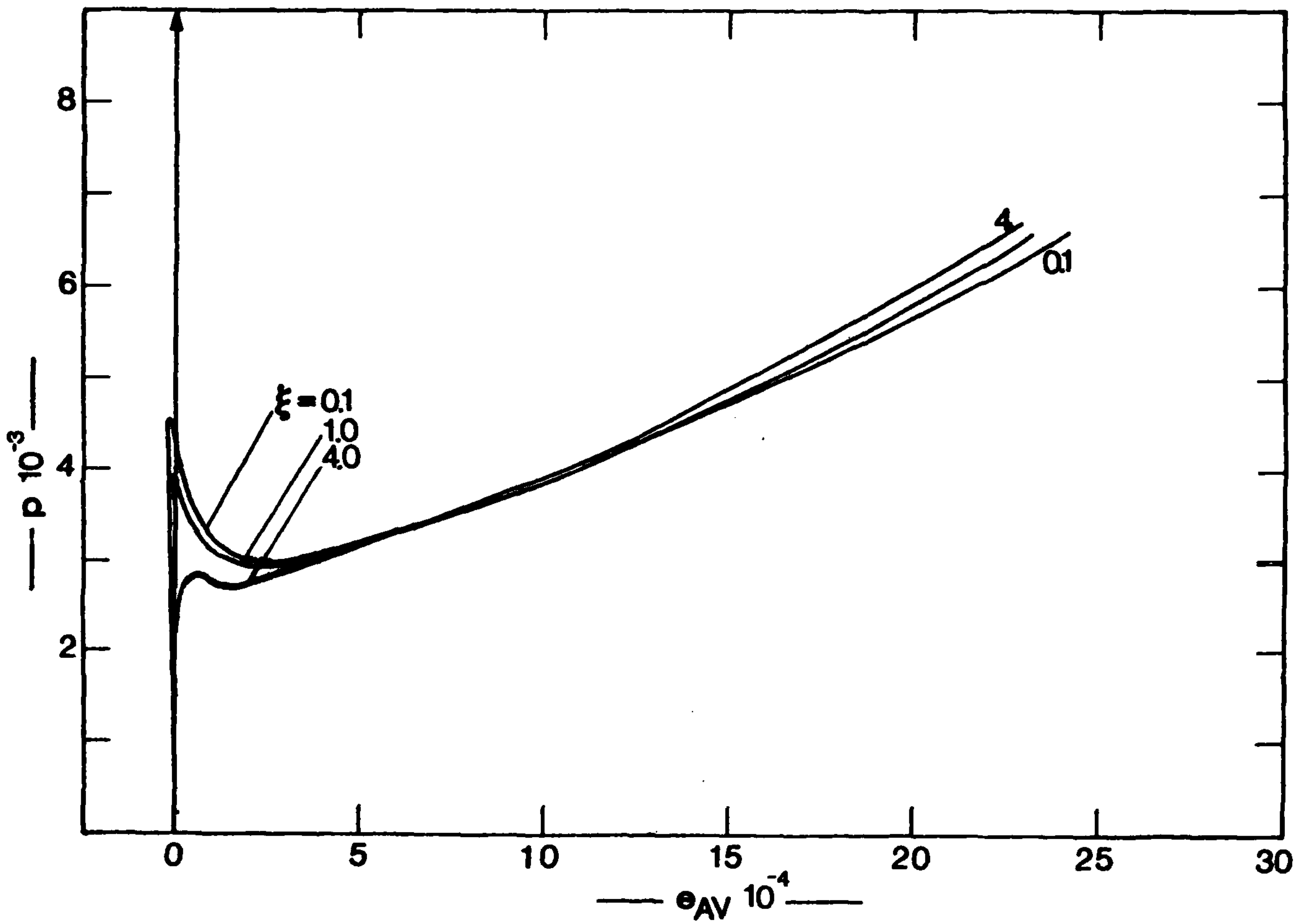

Fig. 3.22 Response of Ring-Stiffened Geometry (pressure load; n = 3)

brium point and motion can escape through this point. (A trajectory can possibly exist for "buckled" motion.)

For the axially-loaded unstiffened geometry, the results are presented both in tabular form (Tables 3.1 and 3.2) and in graphical form (Figs. 3.6, 3.15 and 3.16).

The critical dynamic load is obtained from the static solution. It corresponds to a load (static) for which the unstable (post-limit point curve) equilibrium point yields a value zero for the modified total potential. The expression for the modified total potential is given by [see Eqs. (3.16) and (3.19)]

$$\begin{aligned} U_{T_{mod}} &= U_T - \frac{\beta_1}{E_{xx_p}} \pi RL\bar{N}_{xx}^2 + 2\pi RLa_1\bar{N}_{xx}^2 \\ &= U_T + \pi RLa_1\bar{N}_{xx}^2 \end{aligned} \tag{3.69}$$

It is seen from Tables 1 and 2 that when a limit point exists and the difference between the limit point load and the minimum post-limit point load is distinct, then a clear dynamic critical load exists. On the other hand, if there is no limit point (Examples 17-21 of Table 3.1), there is no critical dynamic load. Similarly, if the value of the limit point load is very close to that of the minimum post-limit point load, it is difficult to have a critical dynamic load (see Examples 35, 36, and 39 of Table 3.1). Fig. 3.6, among others, shows a plot of λ^d versus the imperfection amplitude parameter, ξ. On the basis of the definition of critical dynamic load, λ^d starts from one and decreases to the common value of λ^ℓ and λ^m at $\xi = 4$. Since the static behavior for $\xi > 4$, is not one of limit point instability, then there is no critical dynamic load for these ξ values, according to the concept and criterion discussed at the

beginning of the chapter [see Eq. (3.2)]. On the other hand, if the dynamic response is limited in the space of the displacement components, then a critical load can be defined. This point is discussed in Ref. 1 and in Chapter 5 of the present report.

The effect of L/R is shown on Fig. 3.15. The value of λ^d is very low for L/R equal to one ($\lambda^d = 0.2$) and it increases rapidly with increasing L/R values to $\lambda^d = .48$ at L/R = 10. This, of course, holds true only for $\xi = 1.0$, but a similar behavior is observed for ξ-values for which the static behavior is the same as for $\xi = 1.0$ (see Figs. 3.3 and 3.4; but not 3.5).

For the axially-loaded stiffened configuration, the results are presented on Fig. 3.23. It is seen from this figure that the internally stiffened geometry (under static conditions) is not as sensitive as the externally stiffened one. Moreover, the ratio of the dynamic load to the static (λ^d/λ^ℓ) is higher for the internally stiffened geometry. Note that the results for the internally stiffened geometry do not extend past $\xi = 1.0$. This is so because the static behavior will soon ($\xi = 1.5$ or so) cease to be of the limit point instability. On the other hand, the results for external stiffening extend to $\xi = 4$.

It is seen that the largest difference (or smallest ratio λ^d/λ^ℓ) between the static and dynamic critical loads occurs at $\xi \simeq 1.0$ (see Figs. 3.6 and 3.23), for axially loaded geometries.

For the pressure-loaded ring-stiffened geometry, critical dynamic loads are obtained by setting the modified potential equal to zero [see Eq. (3.2)]. For this case, C, for a given load, denotes the potential associated with the static primary path mode. This means that the corresponding static problem must be solved (without allowing static buckling)

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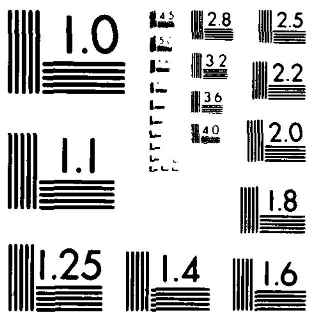

MICROCOPY RESOLUTION TEST CHART

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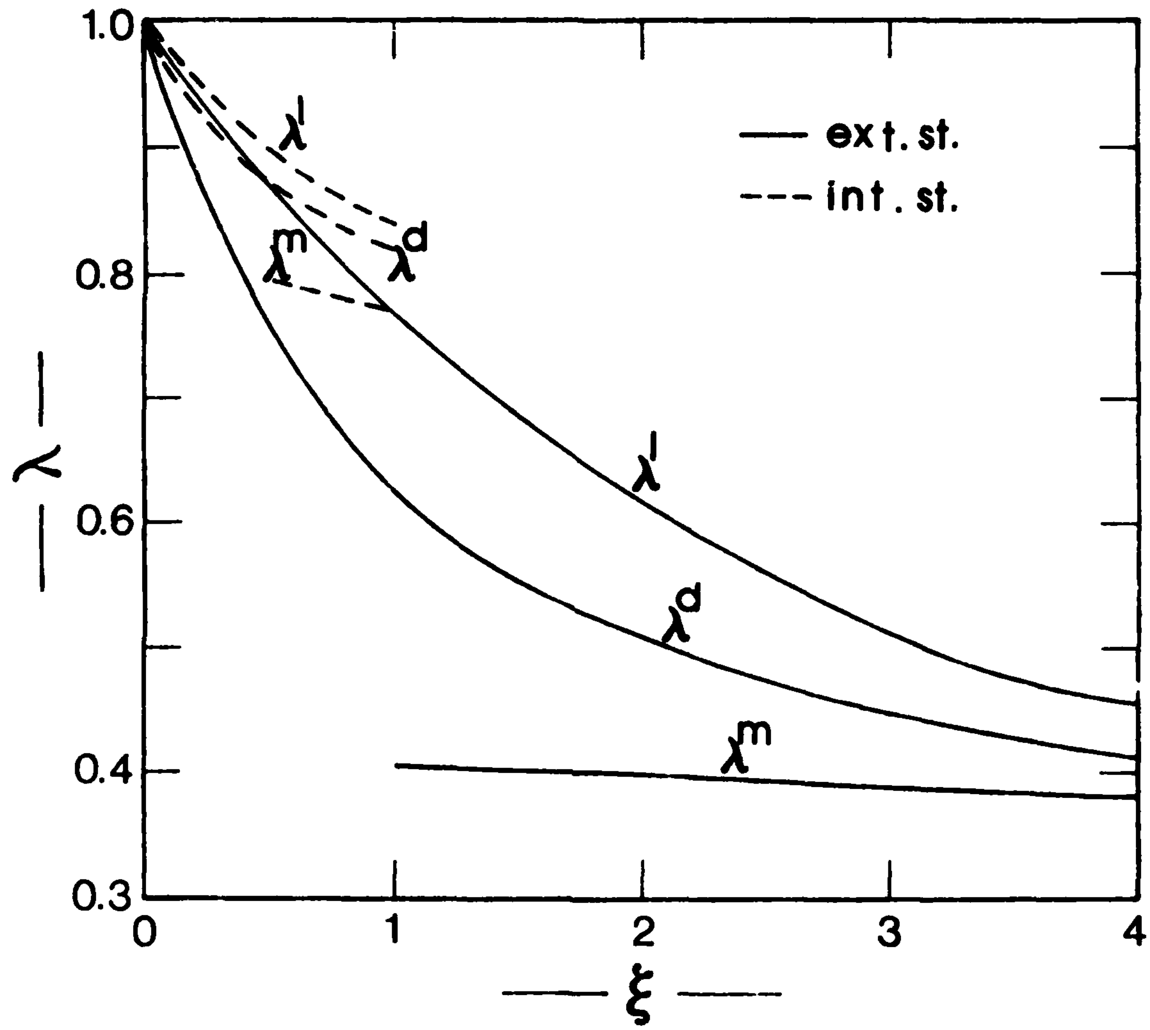


Fig. 3.23 Plots of Load Parameters (λ^{ℓ}; λ^{d}; λ^{m}) versus ξ for the Stiffened Geometries (axial load)

with regard to static application of the pressure and by finding the corresponding axisymmetric displacement (breathing mode). Then at each level of the pressure, the corresponding total potential is calculated and for each value of p this corresponds to the value of C in Eq. (3.2).

The estimated results for dynamic critical pressure along with the limit-point values and the minimum post-limit point values are shown on Table 3.3. Only results corresponding to n = 3 are shown, because this value of n governs static and dynamic building.

Table 3.3 Pressure-Loaded, Ring-Stiffened Cylindrical Shell.

n	ξ	p^{ℓ} psi	p^{d} psi	p^{m} psi	p^{d}/p^{ℓ}
3	0.1	4,500	4,470	3,000	0.9933
3	1.0	3,845	3,790	2,970	0.9857
3	4.0	2,830	2,755	2,740	0.9735

It is seen from these results that the reduction in critical pressure, because of the sudden application (dynamic versus static), is very small. This should not be considered as a general conclusion, but more data need to be generated.

SECTION IV

THE PINNED HALF-SINE ARCH

In this particular chapter the concepts of dynamic stability are applied to a half-sine low arch, subjected to a transverse load with a half-sine spatial distribution. The static analysis of such an arch may be found in Ref. 32. Some dynamic stability aspects of this or similar geometries may be found in Refs. 2, 11, 33, 34 and 35. In this chapter some of these studies are summarized, particularly those of Refs. 2, 11, and 35. These studies include loads of constant magnitude and finite durations (as well as the extreme cases of the duration time approaching zero and infinity), and the study of various effect, all of which are presented in Ref. 1. These include, the effect of static preloading and small damping.

Geometry and Governing Equations

Consider a slender arch of small initial curvature and symmetric cross section. Furthermore, $w_o(x)$ denotes the initial shape of the middle line, $w(x)$ the shape of the middle line after deformation, and $u(x)$ the horizontal displacement of any point of the midplane (see Fig. 4.1). The following nondimensionalization is introduced

$$x = \frac{L}{\pi} \xi$$

$$w(x,t) = \rho \eta (\xi,t) \text{ where } \rho^2 = \frac{I}{A}$$

$$t = \tau \frac{L/\pi}{\sqrt{E\frac{\epsilon_E}{\sigma}}} \text{ where } \epsilon_E = (\pi \rho/L)^2 \tag{4.1}$$

and E is the Young Modulus and $q(\xi,t) = \dfrac{\rho}{AE\epsilon_E^2} Q^* (x,t)$

where $Q^*(x,t)$ denotes the external force per unit length.

The specific problem to be considered in this section consists of a low pinned arch for which $w_o(x)$ and consequently $\eta_o(\xi)$ is a half-sine-wave. The distributed load, which is applied suddenly with constant magnitude for a finite duration, τ_o, is also a half-sine-wave. The initial shape is given by

$$\eta_o(\xi) = e \sin \xi \quad 0 < \xi < \pi \tag{4.2}$$

where e is the initial rise parameter. Since $(w_o)_{max} = \rho e$ and $e = \frac{(w_o)_{max}}{p}$, and if the cross section is rectangular of width b and thickness h, then $p = \frac{h}{2/3}$ and $e = 2/3 \frac{(w_o)_{max}}{h}$, which clearly shows that e is a measure of the ratio of the initial maximum rise to the thickness of the arch.

The expression for the loading is given by

$$q(\xi,\tau) = q_1(\tau) \sin \xi \tag{4.3}$$

The response of the arch, $\eta(\xi,t)$ is represented by

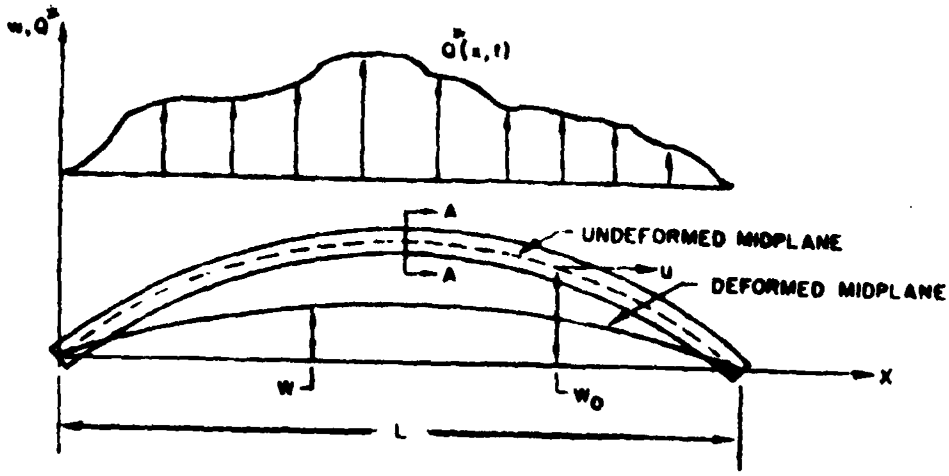
$$\eta(\xi,t) = \eta_o(\xi) + \sum_{n=1}^{2} a_n (t) \sin \eta\xi, \quad 0 < \xi < \pi, \quad r > 0 \tag{4.4}$$

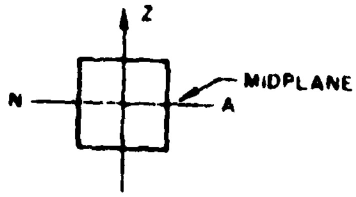
Complete analysis of the problem is given by Simitses in references 11 and 32.

A more convenient set of equations may be obtained by introducing the following new parameters

$$r = a_1 + e \text{ and } p = q_1 + e \tag{4.5}$$

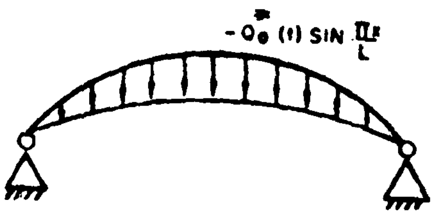
the expression for the nondimensionalized total potential under p





Section A-A

(a) Geometry



(b) The Pinned Arch

Fig. 4.1. The Low Arch; Geometry and Sign Convention

$\left(\bar{U}_T^P = \frac{4U_T^{q_1}}{P_E \epsilon_E L}\right.$ where $U_T^{q_1}$ the total potential under the load q_1 and $P_E =$

$\frac{\pi^2 EI}{L^2}$ denoting the first Euler load$\left.\right)$ is given by

$$\bar{U}_T^P = \frac{1}{8}(r^2 - e^2 + 4\, a_2^2)^2 + r^2 - e^2 + 16a_2^2 + 2p(e-r) \tag{4.6}$$

which is quite similar to the potential energy for the two-degree of freedom model (Model C) discussed in Ref. 1.

Furthermore, neglecting the rotatory and in-plane kinetic energies, the following expression for the nondimensionalized kinetic energy (T is the kinetic energy) is obtained.

$$\bar{T} = \frac{2}{\pi} \int_0^\pi \overset{o}{\eta}{}^2 \, d\xi \tag{4.7}$$

Use of Eq. (4.4) in Eq. (4.7) yields the following expression for $\bar{T}$

$$\bar{T} = \overset{o}{r}{}^2 + \overset{o}{a}{}_2^2 = (1 + a_2'^2)\, \overset{o}{r}{}^2 \tag{4.8}$$

where $(\ ') = \frac{\partial}{\partial r}$ and $(\ ^o) = \frac{d}{d\tau}$

Critical Dynamic Conditions

Dynamic Stability under constant load of finite duration has been discussed in Ref. 1 and through simple mechanical models, criteria and estimates for critical conditions were presented. The same problem is posed here, but applied to a pinned low arch.

The expression for the "zero load" total potential is given by

$$\bar{U}_T^c = \frac{(r-e)^2}{8}(r^2 + 2\,er + e^2 + 8) + a_2^2(2a_2^2 + r^2 + 16 - e^2) \tag{4.9}$$

Through a static stability analysis, the following stationary points on the "zero load total potential" are obtained:

Pt. 1 at $[e,0]$ Stable (Relative min.)

Pt. 2 at $\left[\frac{1}{2}\left(-e + \sqrt{e^2-16}\right), 0\right]$ Unstable (Relative max.)

Pt. 3 at $\left[\frac{1}{2}\left(e + \sqrt{e^2-16}\right), 0\right]$ Stable (Relative min.)

Pt. 4 at $\left[\frac{e}{3}, \sqrt{\frac{2e^2}{9} - 4}\right]$ Unstable (Saddle point)

Pt. 5 at $\left[\frac{e}{3}, \sqrt{\frac{2e^2}{9} - 4}\right]$ Unstable (Saddle point)

It is proven (Ref. 11) that saddle points exist for $e > 4$. For this range of e-values, the "zero load" total potential value at the saddle points, pts. 4 and 5 is smaller than the corresponding value at the relative maximum, pt. 2. On the basis of this observation the motion can possibly become "buckled" through the saddle points, pts. 4 and 5. The corresponding condition for this case is a "possible critical condition". On the other hand, if the imparted energy, by the applied force at the release time, is sufficient to reach the relative maximum (unstable) static equilibrium point, pt. 2, "buckled" motion is guaranteed and the corresponding critical condition is a "minimum guaranteed one". The former is termed sufficient condition for dynamic stability while the latter sufficient condition for dynamic instability by Hsu [6 - 10].

Next, the computational procedure for finding the possible critical condition is outlined.

The stability criterion for this case (see Ref. 1) is expressed by

$$\bar{U}_T^o(\tau_o) - \bar{U}_T^P(\tau_o) \le \bar{U}_T^o(L_u^o) \tag{4.10}$$

where L_u^o is the unstable static equilibrium point under zero load and τ_o the release time. The equality sign refers to a critical condition, while the inequality sign refers to a dynamically stable condition.

Use of Eq. (4.10) for this geometry yields

$$r \Big|_{\tau = \tau_{o_{cr}}} = e - \frac{8\left(\frac{e}{3} - 2\right)}{e - p} \qquad (4.11)$$

where $r_{o_{cr}}$ is the critical release time.

Moreover, for $0 < \tau < \tau_o$ conservation of energy yields (during this time the system is loaded)

$$\bar{U}_T^p + \bar{T}^p = 0 \qquad (4.12)$$

For a given path of motion, integration of Eq. (4.12), yields a relation between the time of release and the position at that instant. Note, that the problem has been cast in the following terms: for a given load, p, find the smallest release time, $\tau_{o_{cr}}$, such that the system may reach an unstable point (saddle point for the minimum possible critical condition) with zero velocity, Eq. (4.11). Since one is interested in obtaining the smallest release time, $\tau_{o_{cr}}$, and since the position at the time of release is path dependent, one can solve the problem by considering the associated brachistochrone problem. The brachistonchrone problem makes use of Eq. (4.12) for this system, and through its solution one obtains the relation between the smallest release time, $\tau_{o_{cr}}$, the position at the instant of release, as well as the path that yields $\tau_{o_{cr}}$. The details of the solution to this brachistochrone problem are similar to the ones presented in Ref. 1. for the two-degree-of-freedom model, Model C. The solution to the brachistochrone problem yields that the minimizing path is characterized by $a_2 \equiv 0$ (symmetric path) and the relation between $\tau_{o_{cr}}$ and the position of the system, r_{cr}, at $\tau_{o_{cr}}$ is

$$\tau_{o_{cr}} = \int_{\sqrt{e}}^{r_{cr}} \frac{dr}{\sqrt{2p(r - e) + e^2 - r^2 - \frac{1}{8}(r^2 - e^2)^2}} \tag{4.13}$$

Computationally, it is simpler for one to assign values of r_{cr} (starting with values close to the initial position, $r = \sqrt{e}$ and $a_2 = 0$), solve for p through Eq. (4.11) and then for $\tau_{o_{cr}}$ through Eq. (4.13).

Note that for the case of the minimum guaranteed critical condition Eq. (4.11) is replaced by a comparable equation which employs the value of the "zero load" total potential at the relative maximum unstable static point.

Numerical results are presented graphically on Figs. 4.2 and 4.3, for the minimum possible critical condition only, and various values of e. The curves of Fig. 4.2 depict critical conditions in terms of applied load, p, versus critical release time, $\tau_{o_{cr}}$. One may observe that as the $\tau_{o_{cr}}$ increases, the corresponding load approaches, asymptotically, the value of p_{cr} for the infinite duration time. Fig. 4.3 presents the same results as Fig. 4.2, but in terms of $(p\tau_o)_{cr}$ versus critical release time $r_{o_{cr}}$. Note that as $\tau_{o_{cr}}$ approaches zero, the value of $(p\tau_o)_{cr}$ approaches that of the critical ideal impulse (see Ref. 11).

Effect of Static Preloading

In evaluating the effect of static preloading, three values are chosen ($e = 5.0, 6.0, 8.0$), and for each e-value the system is initially loaded quasi-statically with a p_o- load smaller than the p_{cr} - static. Then, the system is loaded dynamically. The following values are used in the dynamic analysis.

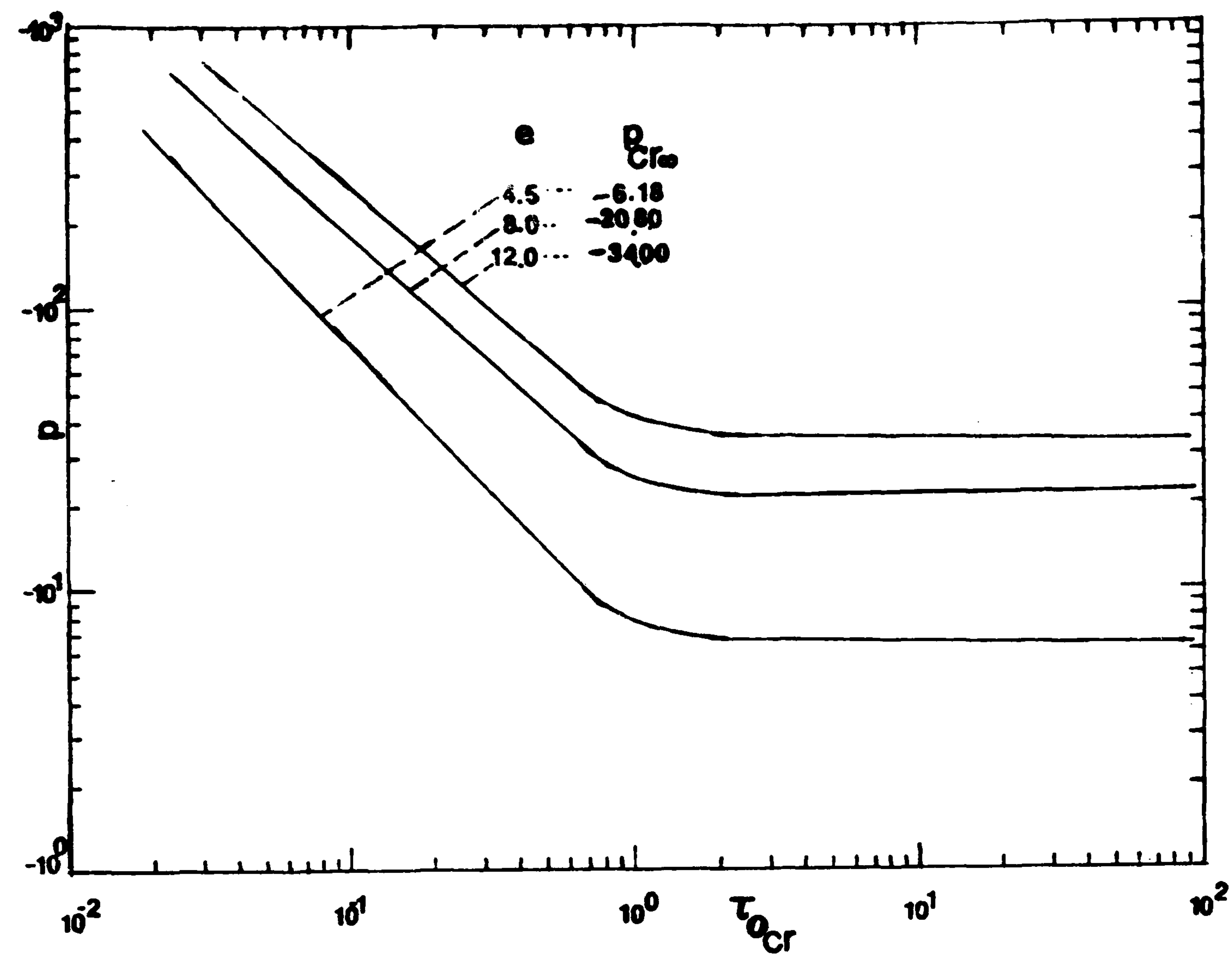


Fig. 4.2. Constant Load, p, versus Critical Duration Time, $\tau_{o_{cr}}$, pinned Arch.

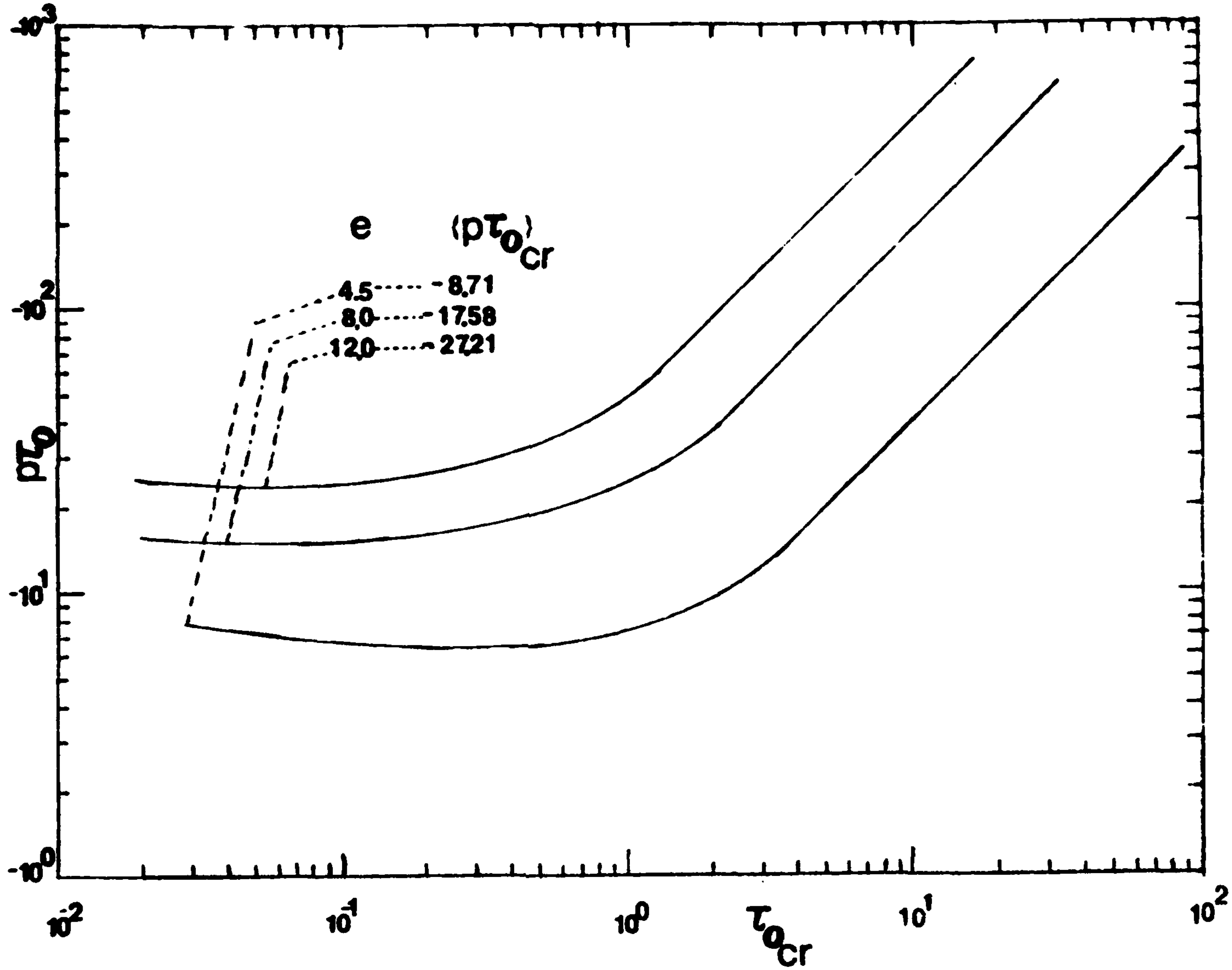


Fig. 4.3. Impulse, $(p\tau_o)$, versus Critical Duration Time, $\tau_{o_{cr}}$, pinned Arch.

$$e = 4.5 \quad ; \quad p_{st_{cr}} = -6.18 \quad ; \quad p_o = -1.0, \; -3.0, \; -5.0$$

$$e = 5.0 \quad ; \quad p_{st_{cr}} = -9.0 \quad ; \quad p_o = -2.0, \; -4.0, \; -6.0$$

$$e = 6.0 \quad ; \quad p_{st_{cr}} = -13.41 \quad ; \quad p_o = -3.0, \; -4.0, \; -6.0$$

First, the extreme cases ($\tau_o \to 0$ and $\tau_o \to \infty$) are analyzed by employing the proper energy equations (see Ref. 1, Section 6).

For example, for the ideal impulse case, the impulse is related to an initial kinetic energy, $\bar{T}_i^{P_o}$(the impulse is imparted into the system as initial velocity) and from conservation of energy (for the preloaded system)

$$\bar{U}_T^{P_o} + \bar{T}_T^{P_o} = \bar{U}_T^{P_o}\,(L_s^{P_o}) + \bar{T}_i^{P_o} \tag{4.14}$$

where $L_S^{P_o}$ is the near static (stable) equilibrium position under P_o (static preloading). Then $\bar{T}_i^{P_o}$is critical (and the corresponding ideal impulse) if the system can reach the unstable static equilibrium point, L_u^P with zero kinetic energy, or

$$\bar{T}_{i_{cr}}^{P_o} = \bar{U}_T^{P_o}\,(L_u^{P_o}) - \bar{U}_T^{P_o}\,(L_s^{P_o}) \tag{4.15}$$

For the second extreme case ($\tau_o \to \infty$), p_{cr} may also be obtained from energy consideration and the criteria developed in Ref. 1. The characteristic equation for this case is obtained from

$$\bar{U}_T^{P_o + P}\left(L_u^{P_o + P}\right) = \bar{U}_T^{P_o + P}\left(L_s^{P_o}\right) \tag{4.16}$$

The ideal impulse, ($p\tau_o$) may be related to the initial kinetic energy $\bar{T}_i^{P_o}$ (in the nondimensionalized form - see Refs. 2 and 11) by the expression

$$(p\tau_o) = - [\bar{T}_i^{P_o}\left(r_s^{P_o}\right)]^{1/2} \tag{4.17}$$

where $r_s^{P_o}$ is the near static (stable) equilibrium position under load P_o.

The critical ideal impulse, through Eqs. (4.17) and (4.15), is obtained by

$$(p\tau_o)_{cr} = - [\bar{U}_T^{P_o}\left(r_u^{P_o}\right) - \bar{U}_T^{P_o}\left(r_s^{P_o}\right)]^{1/2} \tag{4.18}$$

Note that the negative sign on the right hand side of Eqs. (4.17) and (4.18) is present because of the sign convention on the load p (see Fig. 4.1). The expression for the total potential is given by Eq. (4.6).

The numerical results are presented on Table 4. 1.

Table 4.1 - Critical Ideal Impulse, $(p\tau_o)_{cr}$.

e = 4.5		e = 5.0		e = 6.0	
P_o	$(p\tau_o)_{cr}$	P_o	$(p\tau_o)_{cr}$	P_o	$(p\tau_o)_{cr}$
0	-8.71	0	-10.06	0	-12.64
-1.0	-5.11	-2.0	- 4.95	-3.0	- 7.95
-3.0	-3.53	-4.0	- 3.64	-4.0	- 6.80
-5.0	-1.93	-6.0	- 2.28	-6.0	- 5.31
-6.18	0	-9.00	0	-13.41	0

Note that the first row gives the ideal impulse without static pre-loading. Note also that, as the value of p_o approaches the value of the static critical load, the additionally imposed critical impulse tends to zero. This is reflected by the results of the last row (Table 4.1).

The critical load for the case of infinite duration, p_{cr_∞}, is obtained by the following steps, for a given e, p_o combination.

a) Solve the symmetric response equilibrium equation (see Refs. 2, 11, 32), given below, for $r_s^{P_o}$ (near stable position)

$$(r_s^{P_o})^3 - (e^2 - 4)\, r_s^{P_o} = 4\, p_o \tag{4.19}$$

b) The static unstable (saddle) equilibrium positions are characterized by (see Ref. 32)

$$r = -\frac{p_o + p}{3}$$

$$\text{and} \qquad a_2^2 = \frac{1}{4}\left[e^2 - \frac{(p + p_o)}{9} - 16\right] \tag{4.20}$$

c) Eq. (4.16) for this system becomes

$$\frac{1}{8}\,(r^2 + 4a_2^2 - e^2)^2 + r^2 - e^2 + 16a_2^2 + 2\,(p_o + p)\,(e - r)$$

$$= \frac{1}{8}\,(r_s^{P_o} - e^2)^2 + r_s^{P_o^2} - e^2 + 2\,(p_o + p)\,(e - r_s^{P_o}) \tag{4.21}$$

The stimulaneous solution of Eqs. (4.20) and (4.21) yields $r_u^{P_o+P}$ and p_{cr_∞}.

The numerical results for all e, p_o combinations are presented in tabular form on Table 4.2.

Table 4.2 - Critical Dynamic Loads, p_{cr_∞} , (Infinite Duration)

e = 4.5			e = 5.0			e = 6.0		
p_o	p_{cr_∞}	$p_o+p_{cr_\infty}$	p_o	p_{cr_∞}	$p_o+p_{cr_\infty}$	p_o	p_{cr_α}	$p_o+p_{cr_\infty}$
0	-3.7	-3.7	0	-5.20	-5.20	0	-8.8	-8.7
-1.0	-4.05	-5.05	-2.0	-5.54	-7.54	-3.0	-8.61	-11.61
-3.0	-2.54	-5.54	-4.0	-3.90	-7.90	-4.0	-8.02	-12.02
-5.0	-0.99	-5.99	-6.0	-2.24	-8.24	-6.0	-6.77	-12.77
-6.18	0	-6.18	-9.0	0	-9.00	-13.41	0	-13.41

Note that the first row results of Table 4.2 are taken from Ref. 11. The results of the last row reflect the fact that if the system is loaded quasistatically up to the limit point, then the additional suddenly applied load that the system can withstand tends to zero.

Finally, for the case of <u>constant load, p, applied suddenly</u> for a finite duration, τ_o, critical conditions are obtained from the following steps:

a) From the static stability analysis obtain $r_s^{p_o}$, $r_u^{p_o}$, and $a_{2_u}^{p_o}$, for each p_o.

b) Use of the energy balance for this model and load case [see Eq. (6.9) of Ref. 1] yields

$$2p\left(r_{o_{cr}} - r_s^{P_o}\right) = \frac{1}{8}\left({r_u^{P_o}}^2 - e^2 + 4{a_{2_u}^{P_o}}^2\right)^2$$

$$+ {r_u^{P_o}}^2 + 16{a_{2_u}^{P_o}}^2 - \frac{1}{8}\left({r_s^{P_o}}^2 - e^2\right)^2 - {r_s^{P_o}}^2 + 2p_o\left(r_s^{P_o} - r_u^{P_o}\right) \qquad (4.22)$$

where r_{cr} is the position r at the instant of release of the load p ($\tau = \tau_o$). In Eq. (4.22), for a given geometry, e, and static load, P_o, everything is known (p_o, e, $r_s^{P_o}$, $r_u^{P_o}$ and $s_u^{P_o}$) except for p and r_{cr}. Therefore, Eq. (4.22) relates p and r_{cr} at the critical condition.

c) Since $\bar{T}^{P+P_o} = (1 + {a_2'}^2)\left(\frac{dr}{d\tau}\right)^2$, then from Eq. (6.4) of Ref. 1, one may write

$$d\tau = \left[\bar{U}_T^{P_o+P}\left(r_s^{P_o}\right) - \bar{U}_T^{P_o+P}\left(r, a_2\right)\right]^{-\frac{1}{2}} dr \qquad (4.23)$$

Invoking the same techniques as the ones used for the same problem but without static preloading in the previous case, the critical time τ_o is computed on the symmetric path $a_2 \equiv 0$.

Integration from $\tau = 0$ to $\tau = \tau_o$ and use of the expression for the total potential [see Eq. (4.6)] yields

$$\tau_{o_{cr}} = \int_{r_s^{P_o}}^{r_{cr}} \left[\frac{1}{8}\left({r_s^{P_o}}^2 - e^2\right)^2 + {r_s^{P_o}}^2 - \frac{1}{8}\left(r^2 - e^2\right)^2 - r^2\right.$$

$$\left. + 2\left(p + p_o\right)\left(r - r_s^{P_o}\right)\right]^{-\frac{1}{2}} dr \qquad (4.24)$$

Note that Eq. (4.28) also relates r_{cr} to p.

A critical condition is characterized by (p, τ_o) that satifics both equations, Eqs. (4.22) and (4.24) . This means that for a given release time, τ_o, find p_{cr} or for a given p find $\tau_{o_{cr}}$. Computationally, though, it is easier to assign values of r_{cr}, solve for p from Eq. (4.22) and then for the corresponding τ_o from Eq. (4.24).

A computer program has been written for these computations. Values of r_{cr} are assigned, starting with $r_s^{p_o} + \delta r$, where δr is very small, and computing the corresponding values of p and $\tau_{o_{cr}}$ for each δr.

The results are presented graphically on Figs. 4.4 - 4.9 for the three values of e. On the first three figures, critical conditions appear as plots of p versus duration time, $\tau_{o_{cr}}$. Note that as $\tau_{o_{cr}}$ become larger and larger, the corresponding value of p approaches p_{cr_∞} (see Table 4.2). On the last three figures (4.7 - 4.9) critical conditions appear as plots of $(p\tau_o)_{cr}$ versus duration time, $\tau_{o_{cr}}$. On these figures, as $\tau_{o_{cr}} \to 0$, the corresponding value of $(p\tau_o)_{cr}$ approaches the critical ideal impulse (see Table 4.1).

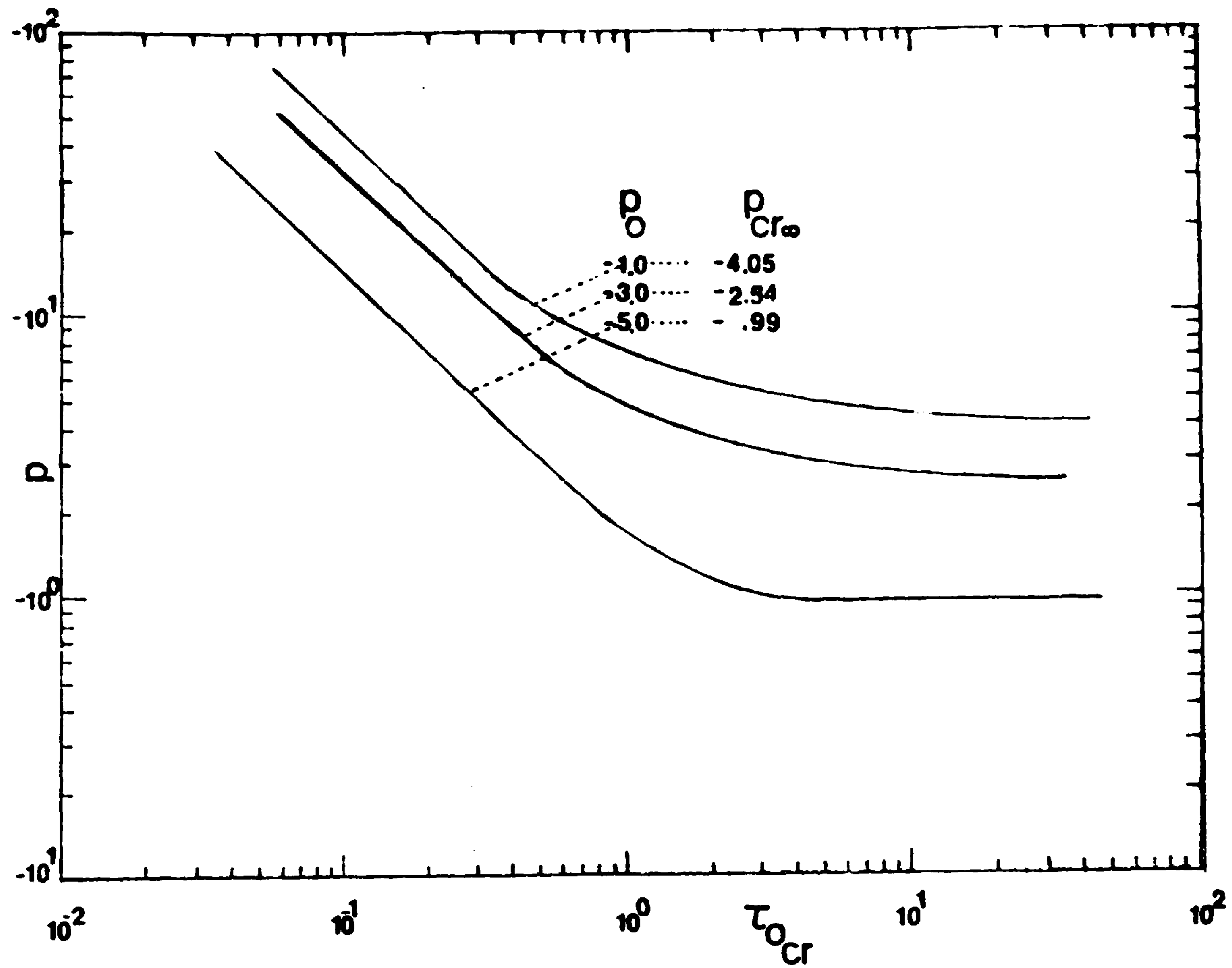


Fig. 4.4 Constant load, p, versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e = 4.5

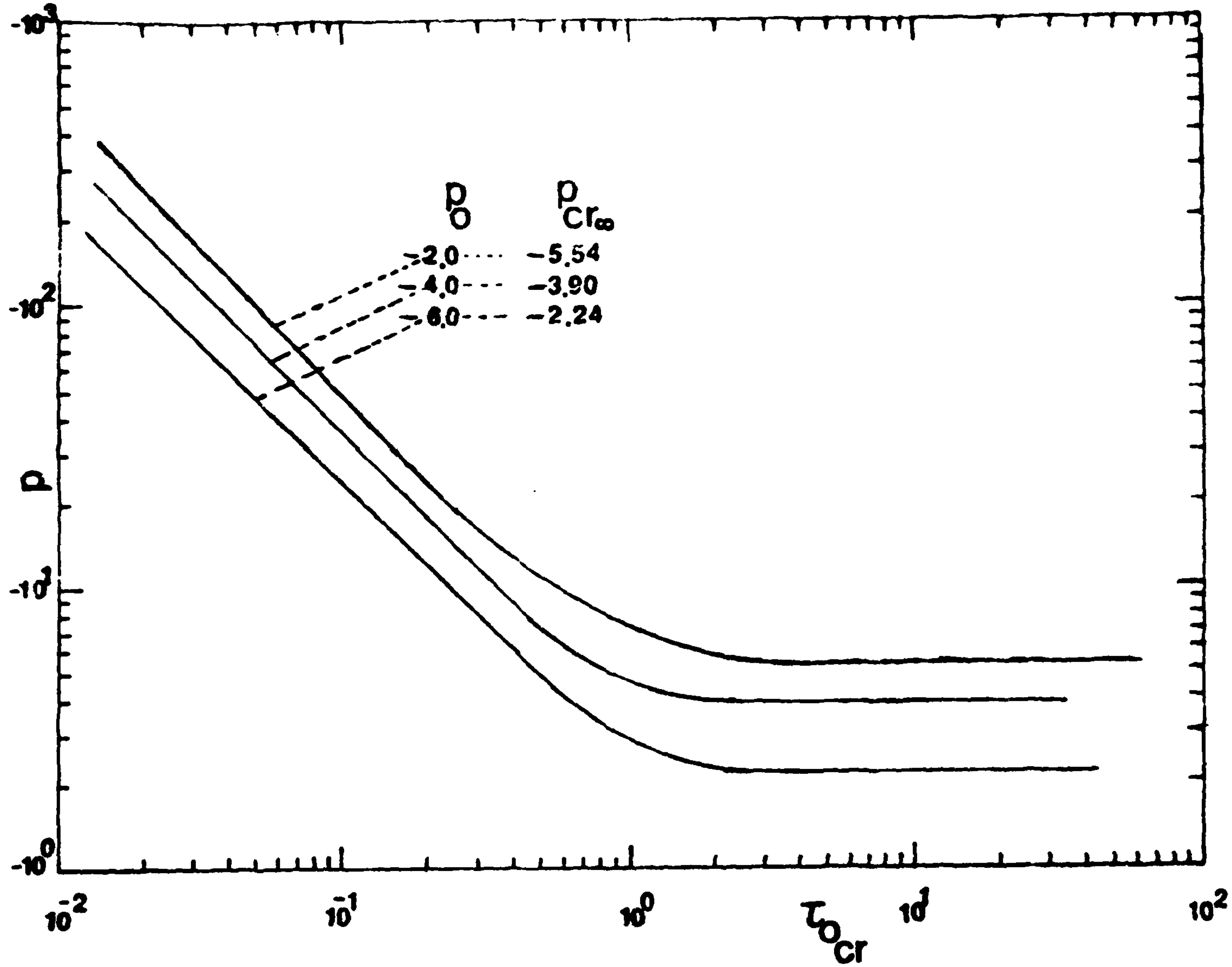


Fig. 4.5. Constant load, p, versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e=5.0

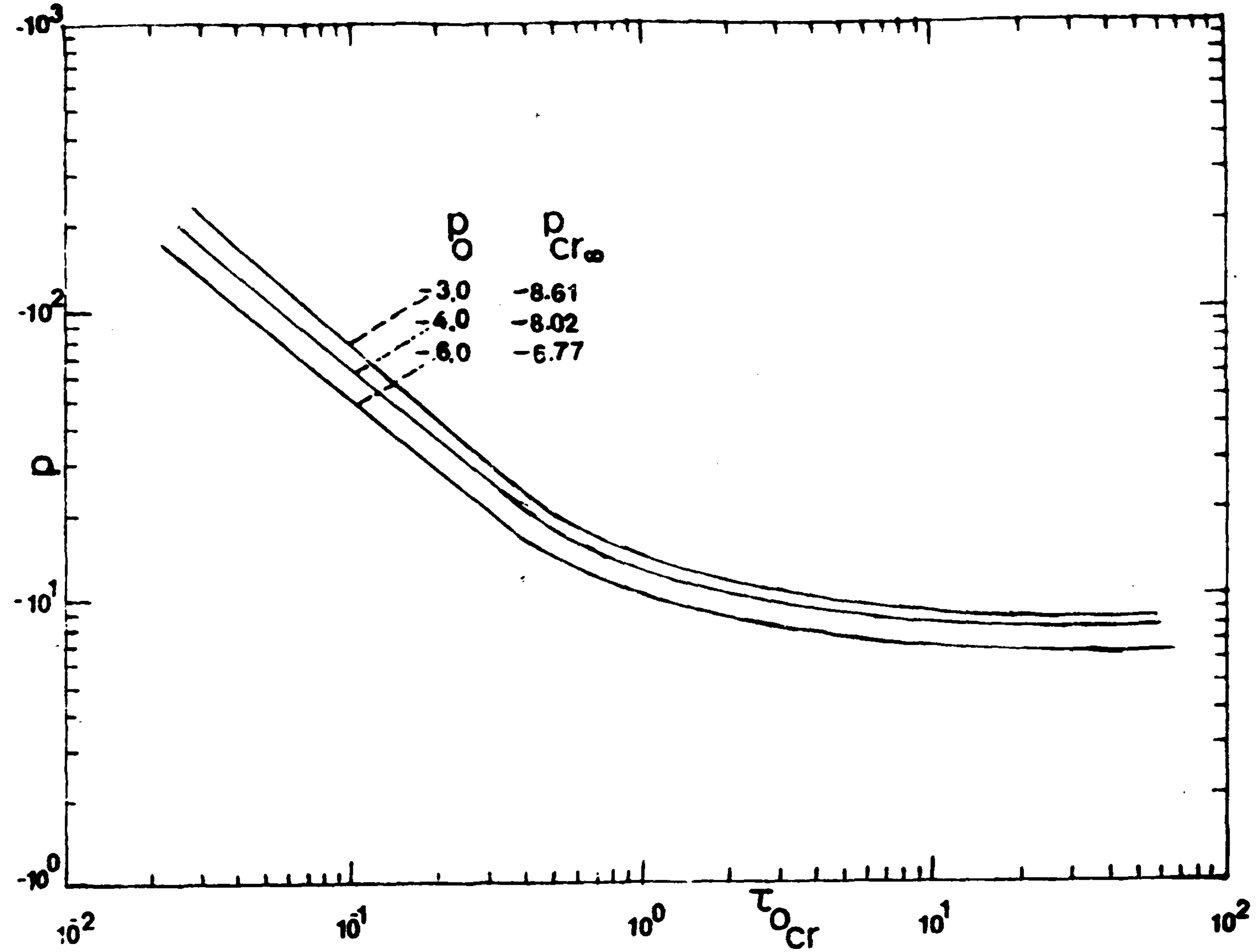


Fig. 4.6. Constant load, p, versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e=6.0

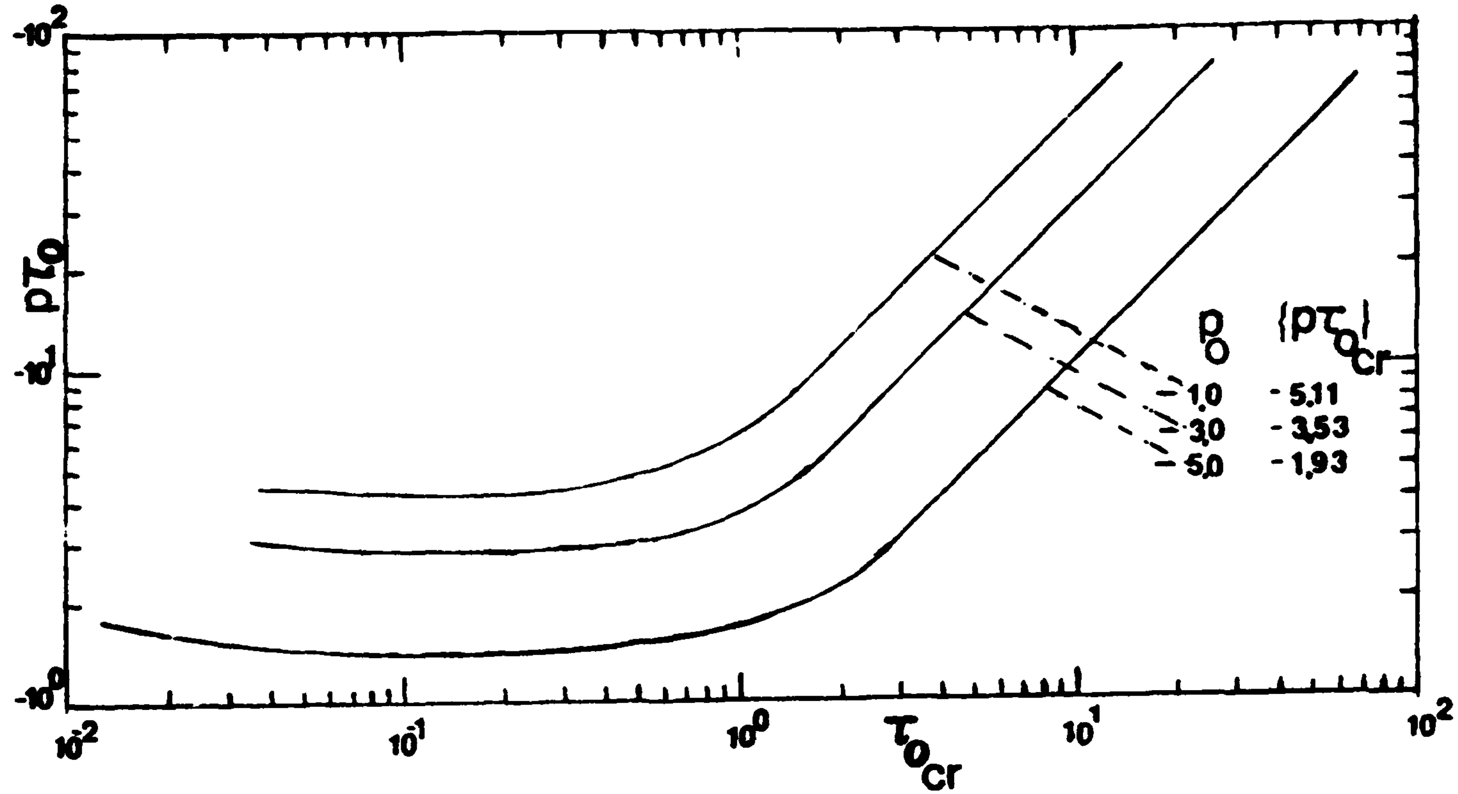


Fig. 4.7. Impulse, $(p\tau_o)$, versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e = 4.5.

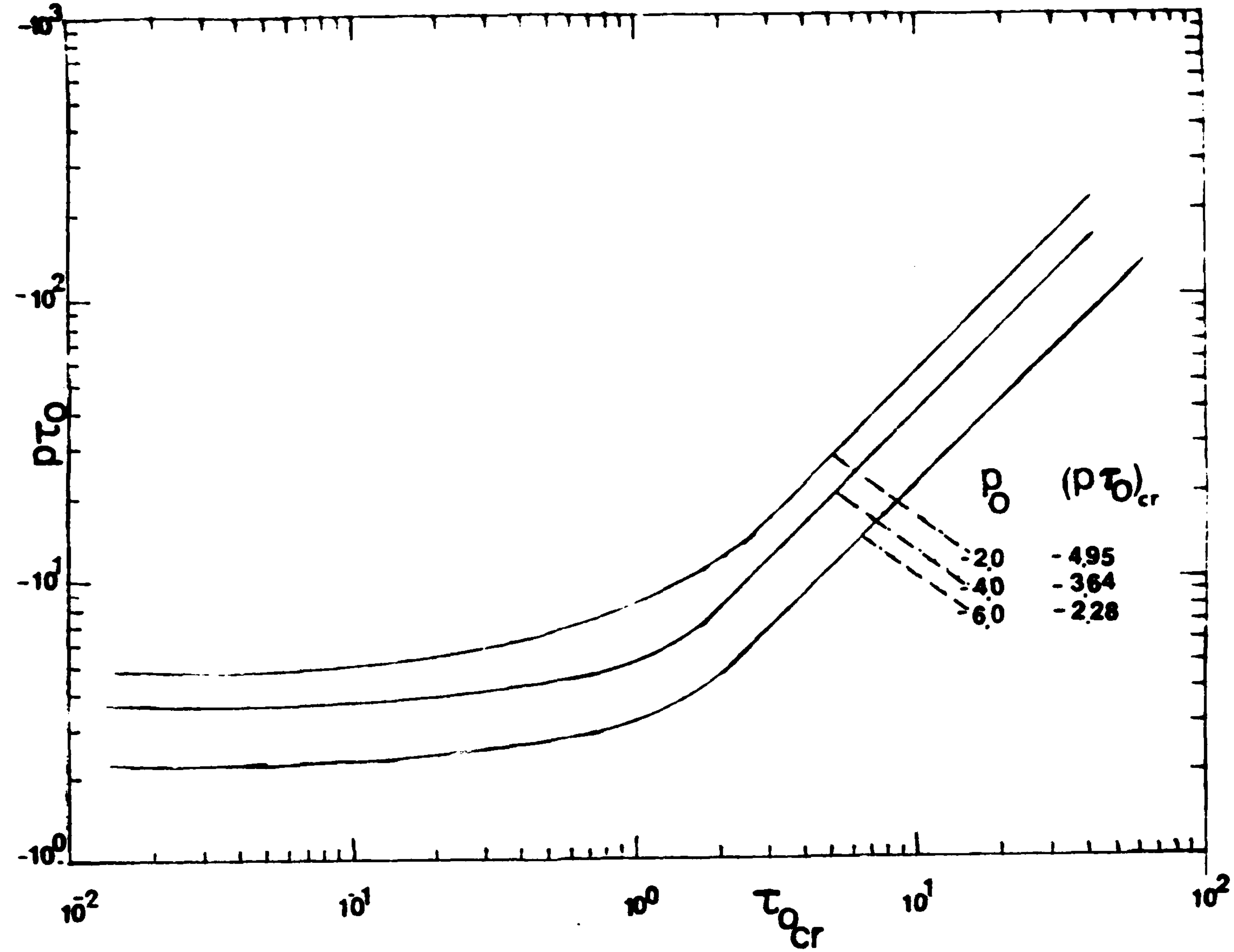


Fig. 4.8. Impulse, $(p\tau_o)$ versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e=5.0

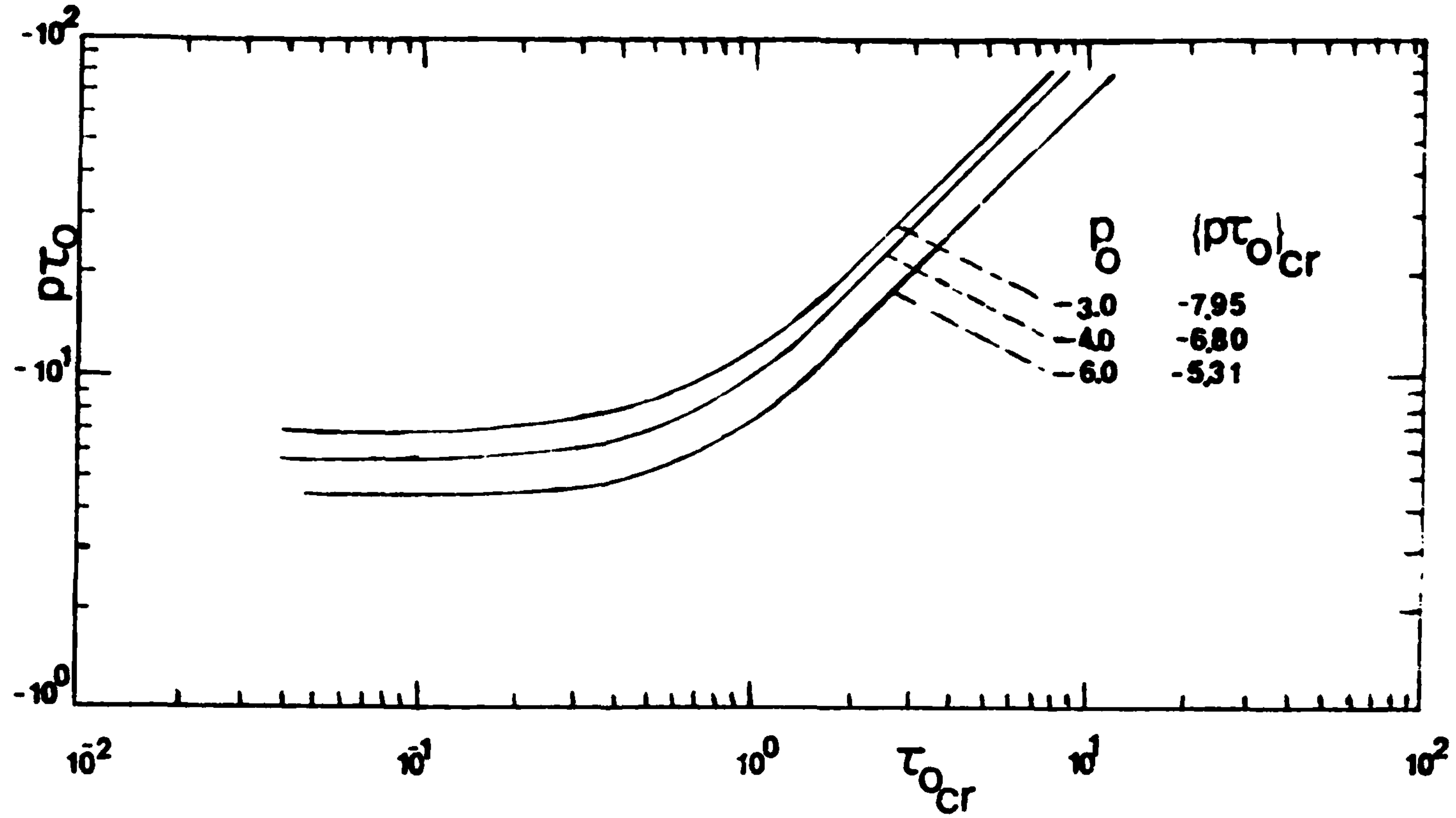


Fig. 4.9. Impulse, $(p\tau_o)$, versus critical Duration Time, $\tau_{o_{cr}}$; Preloaded pinned Arch, e = 6.0.

Effect of Small Damping

In this section, the effect of small damping on the dynamic stability of the arch (subjected to a constant load of finite duration) is investigated. If μ indicates the damping coefficient, the dissipated energy D, because of damping, is given by [see Section VII of Ref. 1 for concepts and details]

$$D = \mu \int_v \int_w \frac{\partial w}{\partial t}\, dw\, dA\, dx = \mu A \int_0^L \int_w \frac{\partial w}{\partial t}\, dw\, dx \tag{4.25}$$

where v stands for volume.

Recalling that $w = \rho\,\{\eta_o(\xi) + r(\tau)\sin\xi + a_2(\tau)\sin 2\xi\}$, where $\rho^2 = \frac{I}{A}$ and $x = \frac{L}{\pi}\xi$, then Eq. (4.25) becomes

$$D = \frac{\mu\rho^2 L}{\pi} A \int_o^\pi \oint_{r,a_2} (\dot{r}\sin\xi + \dot{a}_2 \sin 2\xi)(dr\sin\xi + da_2 \sin 2\xi)\, d\xi \tag{4.26}$$

Since the symmetric path ($a_2 \equiv 0$) is the solution to the undamped system, and since the integrand in Eq. (4.26) must contain only functions of the undamped system, then D reduces to

$$D = \frac{\mu\rho^2 L}{\pi} A \int_e^{r_{cr}} \dot{r}\,dr \int_o^\pi \sin^2\xi\, d\xi = \frac{\mu\rho^2 LA}{2} \int_e^{r^P_{cr}} \dot{r}\,dr \tag{4.27}$$

The nondimensionalization of D is given by

$$\bar{D} = \frac{4D}{P_E\, \epsilon_E\, L} = \bar{\mu} \int_e^{\bar{r}^P_{cr}} \mathring{r}\,dr$$

where $\bar{\mu} = \dfrac{2\mu\rho^2 A \sqrt{E\,\frac{\epsilon_E}{\sigma}}}{P_E\, \epsilon_E\, \frac{L}{\pi}}$ indicates the nondimensionalized damping coefficient,

and $(^{\circ}) = \frac{\partial}{\partial \tau}$ with τ given in Eqs. (4.1).

Since r^P_{cr} is expanded in Taylor's series of $\bar{\mu}$ as

$$r^P_{c_r} = {}_o r^P_{c_r} + \bar{\mu}\ {}_1 r^P_{c_r} + 0\ (\bar{\mu}^2) \tag{4.28}$$

and ${}_o r^P_{c_r}$ stands for the critical r-coordinate for the undamped system. Then Eq. (4.10) yields

$${}_o r^P_{c_r} = e - \frac{8\left(\frac{e^2}{3} - 2\right)}{e - p} \tag{4.29}$$

However, the trajectory that the system follows, from the time of release of the load until it reaches the unstable saddle point $\left[r = -\frac{e}{3}\ ,\ a_2 = \pm \left(\frac{2e^2}{9} - 4\right)^{\frac{1}{2}}\right]$, is unknown. Since Eq. (4.25) gives the dissipated energy during this period of time as

$$\bar{D} = \bar{\mu} \int_{{}_o r_{cr}}^{-\frac{e}{3}} \int_{w} (\sin^2 \xi + a_2' \sin^2 2\xi)\ \overset{o}{r}\ dr\ d\xi$$

then by following the same procedure as in Section VII of Ref. 1 for Model C, a conservative estimate for the critical condition is obtained by assuming a symmetric path ($a_2 \equiv 0$).

Then, from Eq. (7.10) of Ref. 1

$${}_1 r^P_{c_r} = - \frac{\int_e^{-\frac{e}{3}} \overset{o}{r}\, r\ dr}{2\ (\ e - p)} \tag{4.30}$$

Moreover, assuming zero initial conditions, through the equation of motion, Eq. (4.23), along the symmetric path ($a_2 \equiv 0$) for the undamped system, one obtains

$$\overset{o}{r} = - \sqrt{-2\rho \ (e - \hat{r}) + \frac{r^2}{8} (2e - r^2 - 8) - \frac{e^2}{8} (e^2 - 8)} \qquad (4.31)$$

where $\hat{r} = r$ if $r \leq r_{cr}$ and $\hat{r} = r_{cr}$ if $r > r_{cr}$. From Eq. (4.30)

$$_1r^p_{c_r} = - \frac{\int_e^{-\frac{e}{3}} \sqrt{-2p \ (e - \hat{r}) + \frac{r^2}{8} (2e - r^2 - 8) - \frac{e^2}{8}} \, rdr}{2 \ (e - p)} \qquad (4.32)$$

In addition, the critical time $\tau_{o_{cr}}$ may be found through Eq. (7.2) of Ref. 1. Recalling that the kinetic energy is given by Eq. (4.8), the critical time $\tau_{o_{cr}}$ is given by

$$\tau_{o_{cr}} = \int_e^{r_{cr}} - \frac{dr}{\sqrt{-2p \ (e - r) + \frac{r^2}{8} (2e - r^2 - 8) - \frac{e^2}{8} (e^2 - 8) - \bar{\mu} \int_e^r \overset{o}{x} x dx}} \qquad (4.33)$$

Expanding $\tau_{o_{cr}}$ in Taylor's series of $\bar{\mu}$ ($\bar{\mu} << 1$) one may write

$$\tau_{o_{cr}} = {}_o\tau_{o_{cr}} + \bar{\mu} \, {}_1\tau_{o_{cr}} + 0 \ (\mu^2)$$

Note that ${}_o\tau_{o_{cr}}$ is the critical time for the undamped system and it is given by

$${}_o\tau_{o_{cr}} = \tau_{o_{cr \, \mu = 0}} = \int_e^{{}_or_{cr}} - \frac{dr}{\sqrt{-2p \ (e - r) + \frac{r^2}{8} (2e^2 - r^2 - 8) - \frac{e^2}{8} (e^2 - 8)}} \qquad (4.34)$$

Moreover, from Eq. (4.33), one may find the expression for ${}_1\tau_{o_{cr}}$, or

$$
{}_1\tau_{o_{cr}} = \frac{\partial \tau_{o_{cr}}}{\partial\bar{\mu}} \Big| \bar{\mu} = 0 = \frac{1}{2}\int_e^{{}_or_{cr}} - \frac{\int_e^{r} \overset{o}{x}xdx}{\left[-2p(e - r) + \frac{r^2}{8}(2e^2 - r^2 - 8)\ \frac{e^2}{8}(e - 8)\right]^{3/2}}
$$

$$
- \frac{{}_1r_{cr}}{\left[-2p(e - {}_or^p_{cr}) + \frac{{}_or^{p^2}_{cr}}{8}(2e^2 - {}_or^{p^2}_{cr} - 8) - \frac{e^2}{8}(e^2 - 8)\right]^{\frac{1}{2}}} \tag{4.35}
$$

where corrections ${}_1r_{cr}$ and ${}_1\tau_{o_{cr}}$ depend only on undamped system parameters.

The governing equations for finding critical conditions in the presence of small damping ($\bar{\mu} << 1$) are Eqs. (4.29), (4.32), (4.34) and (4.35). These four equations relate the given small damping coefficient $\bar{\mu}$, the applied load p, the time parameters ${}_o\tau_{o_{cr}}$ and ${}_1\tau_{o_{cr}}$, and the position parameters ${}_or^p_{cr}$ and ${}_1r^p_{cr}$. A critical condition is expressed in terms of a load level p and the corresponding time $\tau_{o_{cr}} = {}_o\tau_{o_{cr}} + \bar{\mu}\,{}_1\tau_{o_{cr}}$. Thus, a critical condition may be found by posing the problem as follows: for a given small damping coefficient $\bar{\mu}$ and load level p, find (through the simultaneous solution of the four governing equations) the corresponding critical time parameters, ${}_o\tau_{o_{cr}}$ and ${}_1\tau_{o_{cr}}$, and position parameters, ${}_or^p_{cr}$ and ${}_1r^p_{cr}$. Note that the range of p-values (assigned) must be greater than dynamic critical load for the case of a suddenly applied constant load of infinite duration, without damping. The computational procedure involves the following steps: (a) assign a p-value and compute ${}_or^p_{cr}$ from Eq. (4.29), (b) employ Eq. (4.32) and solve for ${}_1r^p_{cr}$, (c) from Eq. (4.34) solve for ${}_o\tau_{o_{cr}}$, and finally (d) employ Eq. (4.35) and solve for ${}_1\tau_{o_{cr}}$.

A computer program is written to accomplish the solution and numerical results are generated for three values of the arch rise parameter e ($e = 4.5, 8.0, 12.0$). These results are presented on Table 4.3.

Note that, since a critical condition corresponds to a set of p, $\tau_{o_{cr}}$ values, a small damping coefficient $\bar{\mu}$ has a stabilizing effect. This effect, though, is very small. For instance, at the high values of the load p (say for $e = 4.5$, $p = -119.00$) the corresponding value for $\tau_{o_{cr}}$ (if $\bar{\mu} = 0.04$) is $0.098 + 0.0015 = 0.0995$. Remember that the $p - \tau_{o_{cr}}$ curve for the undamped system (see Fig. 4.2) is very steep at the high p-value and virtually flat at the low values of p. On the other hand, when $p = -7.21$ (a value close to $p_{cr_\infty} = -6.18$) the corresponding critical time is $\tau_{o_{cr}} = 0.59 + 0.04 = 0.63$. Since the curve is very flat at this load p-value, one may conclude that the effect of small damping is virtually negligible.

Table 4.3. Incremental Critical Time ${}_1\tau_{o_{cr}}$ ($\bar{\mu}$ = 1) for several Finite Duration Loads, p. (pinned shallow Arch).

e	p	${}_0\tau_{o_{cr}}$	${}_1\tau_{o_{cr}}$
4.50	-119.00	0.098	.337
	- 57.00	0.134	.379
	- 30.00	0.210	.446
	- 20.00	0.250	.517
	- 7.21	0.590	.970
8.00	-658.00	0.032	.450
	-214.00	0.089	.514
	-103.00	0.215	.620
	- 75.00	0.293	.706
	- 39.00	0.607	1.203
12.00	-561.00	0.053	.717
	-274.00	0.092	.840
	-179.00	0.183	.990
	-115.00	0.527	1.380

SECTION V

OTHER SYSTEMS

As explained in Chapter 1, it is possible to extend the concept of dynamic buckling to all structural systems regardless of their behavior under static application of the loads (see Figs. 1.1 - 1.5). This extension is presented in Ref. 1, and it is based on limiting the deflectional response of a structure (when loaded suddenly), which is in agreement with requiring boundedness of deflectional response. One should observe that in limiting the deflectional response, boundedness is automatically satisfied (in some cases enforced), while the reverse is not true.

Some examples are presented in this chapter, in order to clarify this extension of the concept of dynamic stability.

The Mass-Spring System

Consider the mass-spring (linear) system shown on Fig. 5.1. Consider a suddenly applied load, P(t), applied at t = 0. This load may, in general, include the weight (mg). In the case of finite duration, consider the weight to be negligible.

First, the problem of constant load suddenly applied with infinite duration is considered.

For this case, one may write the equation of motion and solve for the response by imposing the proper initial conditions.

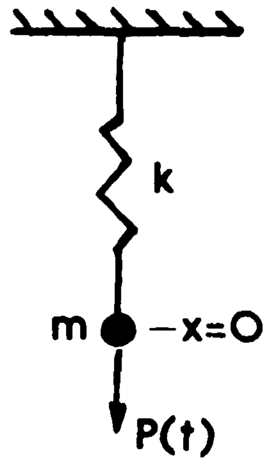


Fig. 5.1 The Mass-Spring System

$$\ddot{x} + \frac{k}{m} x = \frac{P}{m} \qquad (5.1)$$

subject to

$$\dot{x}(0) = x(0) = 0 \qquad (5.2)$$

where the dot denotes differentiation with respect to time.

By changing the dependent variable to

$$y = x + C \qquad (5.3)$$

where C is a constant,

the equation of motion and initial conditions become

$$\ddot{y} + \frac{k}{m} y = 0 \qquad (5.4)$$

$$y(0) = -\frac{P}{k} \quad \text{and} \quad \dot{y}(0) = 0 \qquad (5.5)$$

The solution is

$$y = -\frac{P}{k} \cos \sqrt{\frac{k}{m}}\, t$$

and

$$x = \frac{P}{k}\left(1 - \cos \sqrt{\frac{k}{m}}\, t\right) \qquad (5.6)$$

Note that

$$x_{max} = \frac{2P}{k} \tag{5.7}$$

and it occurs at

$$\sqrt{\frac{k}{m}}\, t = \pi \quad \text{or at} \quad t = \pi \sqrt{\frac{m}{k}} = T/2 \tag{5.8}$$

where T is the period of vibration.

Note that if the load is applied quasistatically, then

$$P_{st} = kx_{st} \tag{5.9}$$

From Eqs. (5.7) and (5.9), it is clear that if the maximum dynamic response, x_{max} and maximum static deflection $x_{st_{max}}$ are to be equal and no larger than a specified value X (deflection limited response) then,

$$P_{st} = 2P_{dyn} \tag{5.10}$$

Because of this, many systems for which the design loads are dynamic in nature (suddenly applied of constant magnitude and infinite duration) are designed in terms of static considerations but with design (static) loads twice as large as the dynamic loads, Eq. (5.10). Note that both loads (P_{st}, P_{dyn}) correspond to the same maximum deflection X.

Next, this same problem is viewed from energy considerations.

First, the total potential, U_T, for the system is given by

$$U_T = \frac{1}{2} kx^2 - Px \tag{5.11}$$

and the kinetic energy, T, by

$$T = \frac{1}{2} m(\dot{x})^2 \tag{5.12}$$

Note that the system is conservative, the kinetic energy is a positive definite function of the velocity (for all t), and that $U_T = 0$ when $x = 0$. Then,

$$U_T + T = 0 \tag{5.13}$$

and motion is possible only in the range of x-values for which U_T is non-positive (see shaded area of Fig. 5.2).

It is also seen from Eq. (5.11) that the maximum x-value corresponds to $2P/k$.

Note that the static deflection is equal to P/k [Eq. (5.9) and pt A on Fig. 5.2]. Therefore, if the maximum dynamic response and maximum static deflection are to be equal to X, then Eq. (5.10) must hold.

Now, one may develop a different viewpoint for this same problem. Suppose that a load P is to be applied suddenly to the mass-spring system with the condition that the maximum deflectional response cannot be larger than a specified value X. If the magnitude of the load is such that

$$\frac{2P}{k} < X \tag{5.14}$$

we shall call the load dynamically subcritical.

When the inequality becomes an equality, we shall call the corresponding load dynamically critical (see Ref. 1). This implies that the system cannot withstand a dynamic load $P > \frac{kX}{2}$ without violating the kinematic constraint. Therefore,

$$P_{dyn_{cr}} = \frac{kX}{2} \tag{5.15}$$

This extension of the energy concept of dynamic stability was first introduced and discussed in Ref. 1.

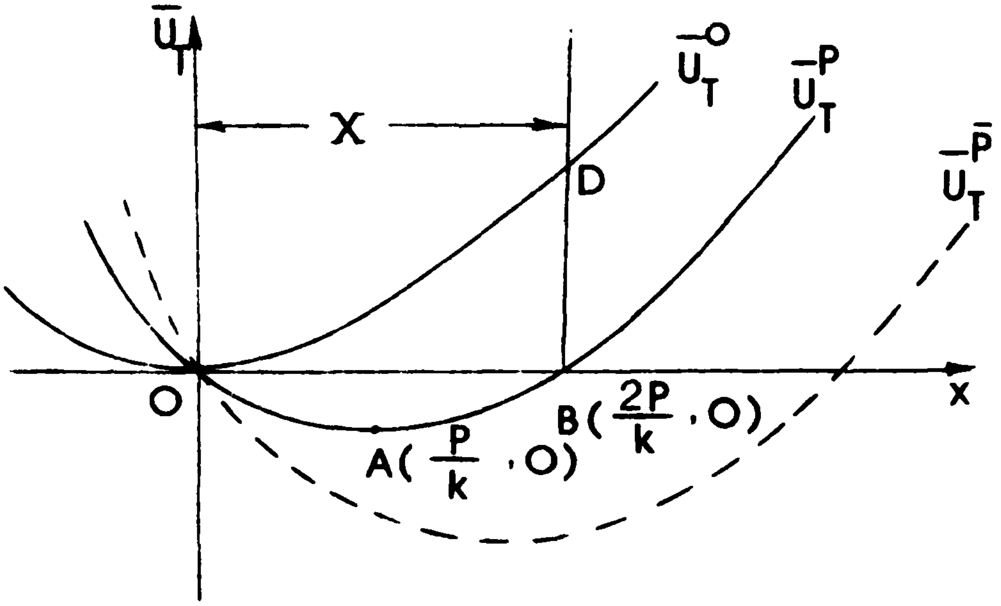


Fig. 5.2 Total Potential Curves (Suddenly Loaded Mass-Spring System).

Moreover, on the basis of this concept, one may find a critical ideal impulse. The question, in this load case, is to find the ideal impulse such that the system response does not exceed a prescribed value X. From Fig. 5.2 and conservation of energy

$$U_T^o + T = T_i \tag{5.16}$$

and T_i is critical if the system can reach position D with zero velocity (kinetic energy). Thus,

$$T_{i_{cr}} = U_T^o(D) = U_T^o(X) \tag{5.17}$$

From the impulse-momentum theorem, the ideal impulse, Imp, is related to the initial velocity and consequently to the initial kinetic energy.

$$\text{Imp} = \lim_{t_o \to 0} (Pt_o) = m\dot{x}_i \tag{5.18}$$

where $\dot{x}_i$ is the initial velocity magnitude (unidirectional case) and t_o is the duration time of a square pulse.

From Eq. (5.18)

$$\dot{x}_i = \frac{\text{Imp}}{m} \tag{5.19}$$

and use of Eqs. (5.12) yields

$$\dot{x}_i = \left(\frac{2T_i}{m}\right)^{1/2} \tag{5.20}$$

Since the critical initial kinetic energy is given by Eq. (5.17), then

$$\text{Imp}_{cr} = (mk)^{1/2}\, X \tag{5.21}$$

Next, the following nondimensionalized parameters are introduced

$$p = \frac{2P}{kX} \; ; \quad \xi = \frac{x}{X} \; ; \quad \tau = t\sqrt{\frac{k}{m}}$$

$$\bar{U}_T = \frac{2U_T}{kX^2} \; ; \quad \bar{T} = \frac{2T}{kX^2} \; ; \quad \overline{Imp} = \frac{2Imp}{X\sqrt{km}} \tag{5.23}$$

On the basis of this Eq. (5.21) becomes

$$\overline{Imp}_{cr} = 2 \tag{5.24}$$

Finally, the concept of dynamic stability is next applied to the general case of a suddenly applied load of constant magnitude but finite duration, t_o. The precise statement of the problem is: find the load, P, for a given duration time, t_o (or vice versa) such that the maximum deflection is no larger than a prescribed value, X. Note that the extreme cases of $t_o \to 0$ and ∞ have been dealt with separately, and that for this case, P must be greater than $P_{dyn_{cr}}$ [see Eq. (5.15)].

For this load case and system, conservation of energy yields

$$U_T^P + T^P = 0 \qquad 0 \le t \le t_o \tag{5.25}$$

and

$$U_T^o + T^o = C \qquad t \ge t_o \tag{5.26}$$

where C is a constant. This constant can be expressed in terms of U_T^P and U_T^o values at the instant of release, t_o. Since there exists kinematic continuity at t_o, $T^P(t_o) = T^o(t_o)$ then

$$C = U_T^o(t_o) - U_T^P(t_o) \tag{5.27}$$

and

$$U_T^o + T^o = U_T^o(t_o) - U_T^P(t_o) \tag{5.28}$$

A critical condition exists if position X can be reached with zero velocity (kinetic energy). Thus, from Eqs. (5.28) and (5.11)

$$\frac{1}{2} kX^2 = Px(t_o) = Px_{cr} \tag{5.29}$$

where x_{cr} is the x-position at the instant of release.

From Eqs. (5.25), (5.11), and (5.12) one may write

$$\frac{1}{2} kx^2 - Px + \frac{1}{2} m(\dot{x})^2 = 0 \quad 0 \le t \le t_o \tag{5.30}$$

or

$$\dot{x} = \left(\frac{2P}{m} x - \frac{k}{m} x^2\right)^{1/2} \tag{5.31}$$

From this one may write

$$dt = \frac{dx}{\left(\frac{2P}{m} x - \frac{k}{m} x^2\right)^{1/2}} \tag{5.32}$$

Integration from zero to $\underline{t_o}$ yields an equation that relates t_o, $x(t_o)$ and P.

$$t_o = \int_0^{x_{cr}} \frac{dx}{\left(\frac{2P}{m} x - \frac{k}{m} x^2\right)^{1/2}} \tag{5.33}$$

Eqs. (5.23) and (5.29) are two equations that relate P, x_{cr}, and t_o. A critical condition is expressed in terms of either P_{cr} for a given t_o or $t_{o_{cr}}$ for a given P-value.

Computationally, it is simpler to assign values of x_{cr} from zero up to X and solve for the corresponding P from Eq. (5.29) and for t_o from Eq. (5.33).

Use of the nondimensionalized parameters, Eqs. (5.23), yields the following system of governing equations

$$p\xi_{cr} = 1$$

and (5.34)

$$\tau_o = \int_0^{\xi_{cr}} \frac{d\xi}{\left(p\xi - \xi^2\right)^{1/2}}$$

Note that the first of Eqs. (5.34) corresponds to Eq. (5.29) and the second to Eq. (5.33). Moreover, the value of ξ_{cr} varies from zero to one.

The simultaneous solution of Eqs. (5.34) yields

$$p = 1/\xi_{cr} \quad \text{and} \quad \tau_{o_{cr}} = \cos^{-1}\left(1 - 2\xi_{cr}^2\right) \tag{5.35}$$

Note that as ξ_{cr} approaches <u>one</u>, $\tau_{o_{cr}}$ is equal to half the period of oscillations and $p = 1$, which is the value that corresponds to the case of constant load suddenly applied, with infinite duration [see Eq. (5.15)].

The results are shown graphically on Fig. 5.3, as plots of p versus t_o/T or $\tau_o/2\pi$.

<u>Parenthesis</u>

Eq. (5.29) may be interpreted in a different way. For instance, one may write

$$kX^2 = 2Px_{cr} \tag{5.36}$$

where x_{cr} is the position of the mass at the instant of release of the force, P, and X is the maximum amplitude of oscillations (maximum dynamic response). Moreover, P/k is a measure of the maximum static displacement, $x_{st_{max}}$ (if P were applied quasi-statically). Then, Eq. (5.36) may be

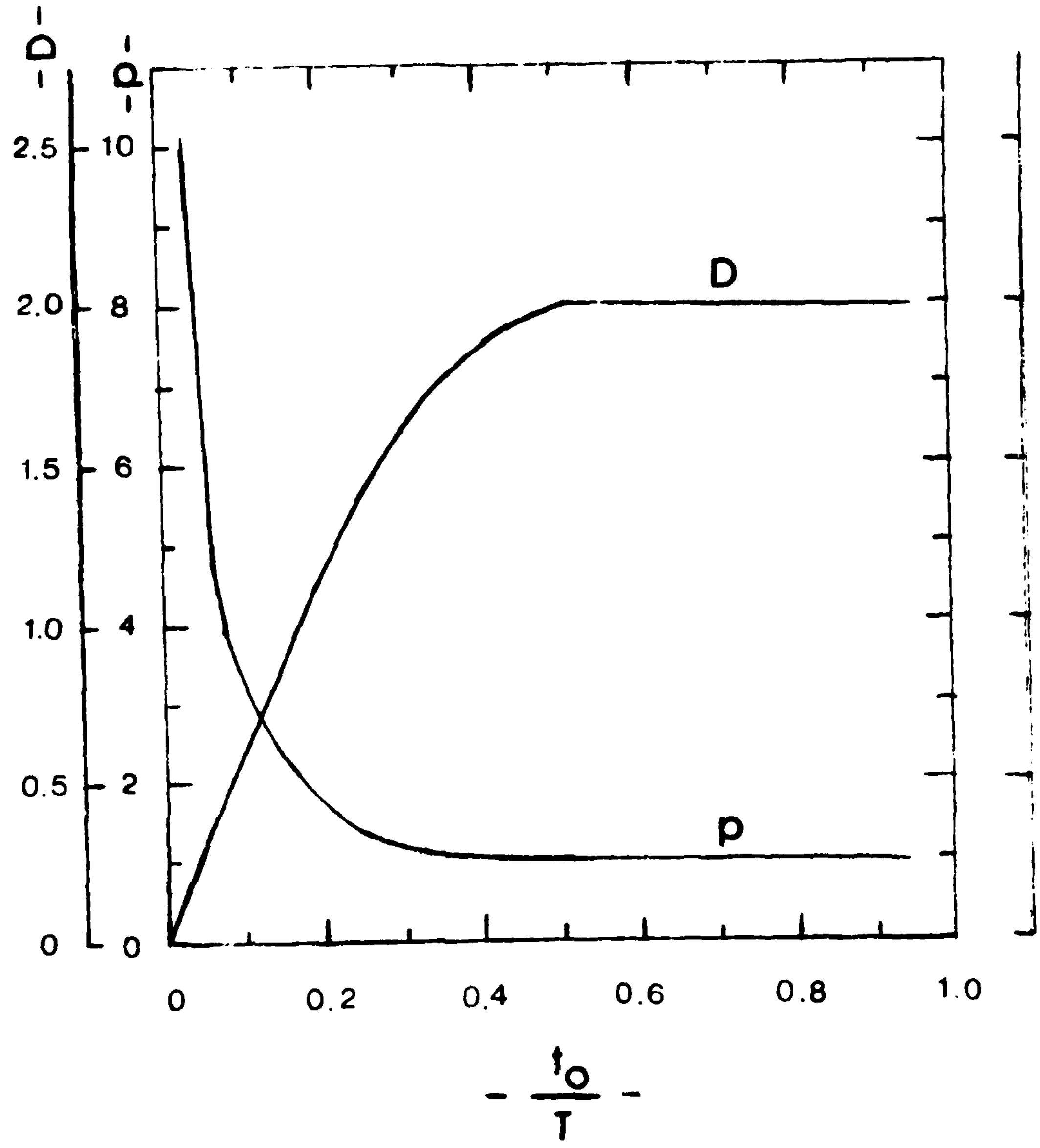


Fig. 5.3 Critical Load and Dynamic Magnification Factor versus Duration time $\left(\frac{t_o}{T}\ \text{or}\ \frac{\tau_o}{2\pi}\right)$.

written as

$$2\,\frac{x_{st_{max}}}{X}\cdot\frac{x_{cr}}{X} = 1 \tag{5.37}$$

Next, $\dfrac{X}{x_{st_{max}}} = D$ is the dynamic magnification factor and $x_{cr}/X = \xi_{cr}$. Therefore, Eq. (5.37) becomes

$$2\xi_{cr}/D = 1 \quad \text{or} \qquad \xi_{cr} = \frac{D}{2} \tag{5.38}$$

Finally, the relation between t_o/T (= $\tau_o/2\pi$) and the magnification factor, D, is obtained from the second of Eqs. (5.35), or

$$\tau_o/2\pi = \frac{t_o}{T} = \frac{1}{2\pi}\cos^{-1}\left(1 - \frac{D^2}{2}\right) \tag{5.39}$$

from which

$$D = 2\sin\left(\frac{\tau_o}{2}\right) = 2\sin\frac{\pi t_o}{T} \tag{5.40}$$

The dynamic magnification factor D, (see p. 94 of Ref. 36) is also plotted on Fig. 5.3 and it is identical to that shown on Fig. 6-6 of Ref. 36.

Note that the parameters plotted on Fig. 5.3 represent two different points of view. The plot of p versus t_o/T depicts the amount of a sudden load with finite duration t_o that corresponds to a maximum amplitude X. On the other hand, the plot of D versus t_o/T shows the magnification of the maximum amplitude (compared to the static one) due to sudden application of the load with duration time, t_o. Note that in both cases, t_o need not be larger than half the period of oscillation or $t_o/T < 2$.

Finally, before closing this section, one can see that the extreme cases of $t_o \to \infty$ and $t_o \to 0$ are special cases of the finite duration case.

From Fig. 5.3, one sees that as $\frac{t_o}{T} \to \infty$ $p \to 1$ which is in agreement with Eq. (5.15). The other extreme case is obtained from Eqs. (5.34). If $t_o \to 0$, then ξ_{cr} is an extremely small number and since $0 < \xi \le \xi_{cr}$ then the second of Eqs. (5.34) becomes

$$\tau_o = \int_0^{\xi_{cr}} \frac{d\xi}{\left(\xi/\xi_{cr} - \xi^2\right)^{1/2}} \simeq \int_0^{\xi_{cr}} \frac{d\xi}{\left(\xi/\xi_{cr}\right)^{1/2}} = 2\xi_{cr} \qquad (5.41)$$

Then

$$(\mathrm{Imp})_{cr} = (p\tau_o)_{cr} = \frac{1}{\xi_{cr}} \cdot 2\xi_{cr} = 2$$

which is identical to the result of Eq. (5.24).

Suddenly Loaded Beams

A large class of structural problems, that may be treated in a similar manner as the mass-spring system, is that of Euler-Bernouilli beams. Under static application of the loads, these configurations exhibit unique stable equilibrium positions at each load level (see Fig. 1.5).

Consider, as an example, the cantilever shown on Fig. 5.4. The load P(t) represents a sudden load with finite duration, in general.

If one assumes that the amplitude of the beam, at any point, is given by the static deflection curve multiplied by a time dependent coefficient, y(t),

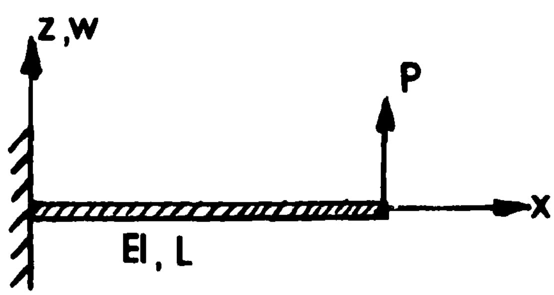


Fig. 5.4 The Cantilever Beam

then

$$w(x,t) = \frac{1}{2}\, y(t) \left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] \qquad (5.42)$$

Note that under static application of the load P, y is the maximum (tip) deflection and it is related to the load by

$$y = \frac{PL^3}{3EI} \qquad (5.43)$$

The total potential for this case is

$$U_T = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 w}{\partial x}\right)^2 dx - PW(L,t) \qquad (5.44)$$

or

$$U_T = \frac{3EI}{2L^3}\, y^2 - Py \qquad (5.45)$$

Note that the stiffness, k, at the free end becomes $k = 3EI/L^3$. With this value for the stiffness, k, Eq. (5.45) is identical to Eq. (5.11) or

$$U_T = \frac{1}{2}\, ky^2 - Py \qquad (5.46)$$

Moreover, the kinetic energy for the cantilever problem is

$$T = \frac{1}{2}\int_0^L \rho\left(\frac{\partial w}{\partial t}\right)^2 dx \qquad (5.47)$$

where ρ is the linear mass density. Substitution of Eq. (5.42) into Eq. (5.47) yields

$$T = \frac{1}{2}\left(\frac{33\rho L}{140}\right)\dot{y}^2 \qquad (5.48)$$

which is similar to Eq. (5.12), provided that $m = 33\rho L/140$. Eq. (5.48) indicates that for the assumed deflection curve, the continuous beam is equivalent to a spring-mass system with k and m given by

$$k = 3EI/L^3 \quad \text{and} \quad m = 33\rho L/140 \tag{5.49}$$

Moreover, the continuous beam is equivalent to a weightless beam with a concentrated mass of m units at the end (see Example 1.5-3 of p. 19 of Ref. 37).

On the basis of the above analogy, the results of the spring-mass system are applicable to the cantilever. In summary, for a prescribed maximum tip deflection, Y, the various critical conditions are given by

$$\begin{aligned} Imp_{cr} &= \lim_{t_o \to 0} (Pt_o) = (mk)^{1/2}\, Y \\ &= \left[0.70714 \frac{EI}{L^2} \rho\right]^{1/2} Y \end{aligned} \tag{5.49}$$

$$P_{dyn_{cr}} = \frac{kY}{2} = \frac{3EI}{2L^3} Y \tag{5.50}$$

Finally, for the case of suddenly applied loads of constant magnitude and finite duration, the results of Fig. 5.3 are applicable provided that the proper expression for p is used.

According to Eqs. (5.23), p may be defined as

$$p = \frac{2P}{kY} = \frac{2PL^3}{3EIY} = \frac{P}{P_{dyn_{cr}}} \tag{5.51}$$

Note also that the magnification factor, D, in this case is the maximum dynamic amplitude ($\bar{Y}$) divided by the maximum static response.

The Imperfect Column

The imperfect column, under sudden application of an axial load, typifies structural systems with static behavior shown on Fig. 1.1. Note that such a system, when of perfect geometry, is subject to bifurcational

buckling with stable post-buckling behavior (smooth buckling). On the other hand, if there exists an initial geometric imperfection (small initial curvature), the system exhibits a unique stable equilibrium path. Moreover, this system has received the most attention, as far as dynamic buckling is concerned when loaded axially either by sudden loads or by time-dependent loads. Two complete reviews (with respect to their date of publication) of this problem may be found in Refs. 38 and 39. As mentioned in these references, the problem dates back to 1933 with the pioneering work of Koning and Taub (Ref. 40), who considered a simply supported, imperfect (half-sine wave) column subjected to an axial sudden load of specified duration. In their analysis, they neglected the effects of longitudinal inertia, and they showed that for loads higher than the static (Euler load) the lateral deflection increases exponentially, while the column is loaded, and after the release of the load, the column simply oscillates freely with an amplitude equal to the maximum deflection. Many investigations followed this work with several variations. Some included inertia effects, others added effect of transverse shear, etc. The real difficulty of the problem, though, lies in the fact that there was no clear understanding by some inveatigators of the concept of dynamic stability and the related criteria.

According to Ref. 39, definition of a dynamic buckling load is possible only if there are initial small lateral imperfections in the column. Instability stems then from the growth of these imperfections. "Buckling occurs when the dynamic load reaches a critical value, associated with a maximum acceptable deformation, the magnitude of which is defined in most studies quite arbitrarily." There is some truth to this, primarily because the elastic column does not exhibit limit point instability or any

other violent type of buckling under static application of the load. There is need for a cautioning remark to the above statement, though. Analytically it has been shown (see Ref. 50) that, if a perfect column is suddenly loaded in the axial direction, the fundamental state is one of axial wave propagation (longitudinal oscillations). For some combination of the structural parameters, this state can become unstable and transverse vibrations of increasing amplitude are possible. Therefore, for this perfect column, there exists a possibility of parametric resonance, which is one form of dynamic instability. In spite of this, mostly all columns are geometrically imperfect and therefore, it is reasonable to investigate the dynamic behavior of imperfect columns including all variations of different effects as reported in Refs. 38-49. These effects include: axial inertia, rotatory inertia, transverse shear, and various loading mechanisms. Moreover, experimental results have been generated to test the various theories and effects.

Finally, the criterion employed in Ref. 39, is the one developed by Budiansky and Roth (Ref. 3), and it is applicable only to imperfection sensitive structural systems, such as shallow arches, shallow spherical caps, and axially-loaded, imperfect, cylindrical shells. The reason that the application of the Budiansky-Roth criterion can possible yield reasonable results for imperfect columns lies in the fact that the corresponding perfect configuration (column) possesses a very flat post-buckling branch. This means that the corresponding imperfect column can experience, at some level of the sudden load or impulse, very large amplitude oscillations (change from small to large amplitude oscillations). Note that the static curve for the imperfect column (static equilibrium), if the load is plotted versus the maximum lateral deflection, yields small values

for the maximum deflection for small levels of the load. As the load approaches the Euler load, the value of the corresponding maximum deflection increases rapidly. On the other hand, if the criterion were to be applied to an imperfect flat plate, it is rather doubtful that reasonable, or any, answers could be obtained. This is so because the slope of the static postbuckling curve, for the perfect plate, is positive, and the imperfect plate exhibits a continuous bending response with smoothly increasing amplitude.

Next, the concept of dynamic stability, as developed in Ref. 1 and discussed in Chapter 1, is applied to an imperfect column. Consider the column shown on Fig. 5.5. The length of the column is L (distance between supports), the bending and extensional stiffnesses are uniform, EI and EA, and the sudden load, P(t), is acting along the horizontal, x, direction. Let u be the horizontal displacement component and $w-w^{\circ}$ the vertical (transverse) displacement component. For the analysis presented herein, the initial geometric imperfection, w°, is a half-sine wave, or

$$w^{\circ}(x) = w_o \sin \frac{\pi x}{L} \tag{5.52}$$

The kinematic relations and the relations between the axial force, P, and bending moment, M, on one hand and the reference axis strain, ϵ°, and change in curvature, κ, on the other are

$$\epsilon = \epsilon^{\circ} + z\kappa \tag{5.53}$$

$$\begin{aligned} \epsilon^{\circ} &= \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - \frac{1}{2}\left(\frac{dw^{\circ}}{dx}\right)^2 \\ \kappa &= -\left(\frac{d^2w}{dx^2} - \frac{d^2w^{\circ}}{dx^2}\right) \end{aligned} \tag{5.54}$$

$$P = EA\epsilon^{\circ} \; ; \quad M^* = EI\kappa \tag{5.55}$$

Moreover, the total potential, U_T^*, expression for the system, is given by (for details, see Ch. 7 of Ref. 32)

$$U_T^* = \int_0^L \left[\frac{P^2}{2EA} + \frac{M^{*2}}{2EI}\right]dx + \bar{P}u(L) \qquad (5.56)$$

Furthermore, the same nondimensionalization as in Ref. 32 is employed, or

$$\xi = \frac{\pi x}{L} \; ; \quad \eta(\xi) = \frac{w(x)}{\rho} \; ; \quad v(\xi) = \frac{u(x)}{\rho} \; ;$$
$$p = \frac{P}{P_E} \; ; \quad M = \frac{M^*}{\rho P_E} \; ; \quad U_T = \frac{4U_T^*}{P_E \bar{\epsilon}_E L} \qquad (5.57)$$

where

$$\rho^2 = \frac{I}{A} \; ; \quad P_E = \frac{\pi^2 EI}{L^2} \; ; \quad \text{and} \quad \bar{\epsilon}_E = \left(\frac{\pi\rho}{L}\right)^2 \qquad (5.58)$$

With these nondimensionalized parameters, one may write

$$p = \frac{1}{2}\left[2\,\frac{v'}{\bar{\epsilon}_E^{\,1/2}} + \left(\eta'\right)^2 - \left(\eta_o'\right)^2\right] \qquad (5.59)$$

$$M = -\left(\eta'' - \eta_o''\right) \qquad (5.60)$$

$$U_T = \frac{1}{\pi}\int_0^{\pi} \left[\frac{1}{2}\left\{2\,\frac{v'}{\bar{\epsilon}_E^{\,1/2}} + \left(\eta'\right)^2 - \left(\eta_o'\right)^2\right\} + 2\left(\eta'' - \eta_o''\right)^2\right]$$
$$+ \frac{4}{\pi\bar{\epsilon}_E^{\,1/2}}\,\bar{p}\,v(\pi) \qquad (5.61)$$

where $(\;)' = \frac{d}{d\xi}$

From Eq. (5.59), one may write

$$v' = -\bar{\epsilon}_E^{\,1/2}\left[\frac{1}{2}\left\{\left(\eta'\right)^2 - \left(\eta_o'\right)^2\right\} + \bar{p}\right] \qquad (5.62)$$

and

$$\int_0^\pi v'd\xi = v(\pi) = -\bar{\epsilon}_E^{1/2}\left[\frac{1}{2}\int_0^\pi \left[\left(\eta'\right)^2 - \left(\eta_o'\right)^2\right]d\xi + \bar{p}\pi\right] \qquad (5.63)$$

Use of Eq. (5.63) yields the following expression for the total potential,

$$U_T = \frac{1}{\pi}\int_0^\pi 2\bar{p}^2 d\xi + \frac{2}{\pi}\int_0^\pi \left(\eta'' - \eta_o''\right)^2 d\xi$$
$$- \frac{4\bar{p}}{\pi}\left[\frac{1}{2}\int_0^\pi \left[\left(\eta'\right)^2 - \left(\eta_o'\right)^2\right]d\xi + \bar{p}\pi\right] \qquad (5.64)$$

Note that in obtaining Eqs. (5.62) and (5.64), use of in-plane static equilibrium is made, or

$$p = \text{const} = -\bar{p} \qquad (5.65)$$

For the dynamic case, this implies that the effect of in-plane inertia is being neglected.

Next, let us consider the case of a suddenly loaded (by an axial force) half-sine (imperfect) column. Then

$$w^o = w_o \sin\frac{\pi x}{L} \qquad \eta_o = \frac{w_o}{\rho}\sin\xi = e\sin\xi \qquad (5.66)$$

Let the response, η, be the form

$$\eta = \left[A(\tau) + e\right]\sin\xi \qquad (5.67)$$

The implication here is that at time $t = 0$ $\eta = e\sin\xi$.

Use of Eqs. (5.66) and (5.67) into the expression for the total potential, Eq. (5.64), yields

$$\bar{U}_T = -2\bar{p}^2 + A^2 - \bar{p}\left(A^2 + 2eA\right) \qquad (5.68)$$

A modified potential, $\overline{U}_{T_{mod}}$, is introduced, such that, regardless of the level of the applied load, the total potential is zero (modified) when $t = 0$ or when $A(0) = 0$. From Eq. (5.68), it is clear that

$$\overline{U}_{T_{mod}} = \overline{U}_T + 2\overline{p}^2 = A^2 - \overline{p}\left(A^2 + 2eA\right) \tag{5.69}$$

The modified total potential is shown graphically on Fig. 5.6 for $\overline{p} = 0$ and $\overline{p}$ = specified value.

As in the case of mass-spring system (note the similarity), critical dynamic conditions can be established, if the maximum allowable amplitude is specified as X.

Only the case of a suddenly applied load of constant magnitude and infinite duration is presented herein. From Fig. 5.6, it is clear that p_{cr_∞} (load for which the system will not exceed the maximum allowable displacement, $A(t) \leq X$) is given by

$$X = \frac{2e\overline{p}_{cr_\infty}}{1-\overline{p}_{cr_\infty}} \quad \text{or} \quad \overline{p}_{cr_\infty} = \frac{X}{X+2e} \tag{5.70}$$

On the other hand, the static load required, such that the maximum static deflection does not exceed the value X, is

$$p_{cr_{st}} = \frac{X}{X+e} \tag{5.71}$$

Note that the above expressions, Eqs. (5.70) and (5.71) hold for $e \neq 0$.

The ratio, ρ^d, of $\overline{p}_{cr_\infty}$ to $\overline{p}_{cr_{st}}$ is given by

$$\rho^d = \frac{\overline{p}_{cr_\infty}}{\overline{p}_{cr_{st}}} = \frac{X+e}{X+2e} = \frac{1+(e/X)}{1+2(e/X)} \tag{5.72}$$

This result is shown graphically on Fig. 5.7.

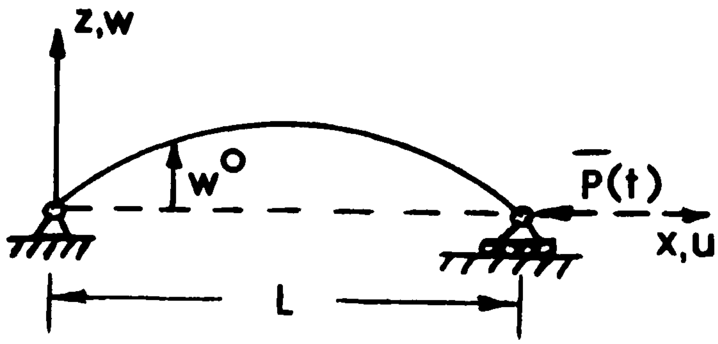


Fig. 5.5 The Imperfect Column

Note that, for very small values of e/X, the ratio ρ^d, is close to 1. For e/X equal to one, $\rho^d = 2/3$. Finally, as e/X becomes very large, then ρ^d approaches the value of one-half.

<u>Parenthesis</u>

If load $\bar{p}$ is applied quasi-statically the maximum deflection, $A_{st_{max}}$, is

$$A_{st_{max}} = \frac{e\bar{p}}{1-\bar{p}} \tag{5.73}$$

If load $\bar{p}$ is applied suddenly the maximum amplitude, $A_{d_{max}}$, is given by

$$A_{d_{max}} = \frac{2e\bar{p}}{1-\bar{p}} \tag{5.74}$$

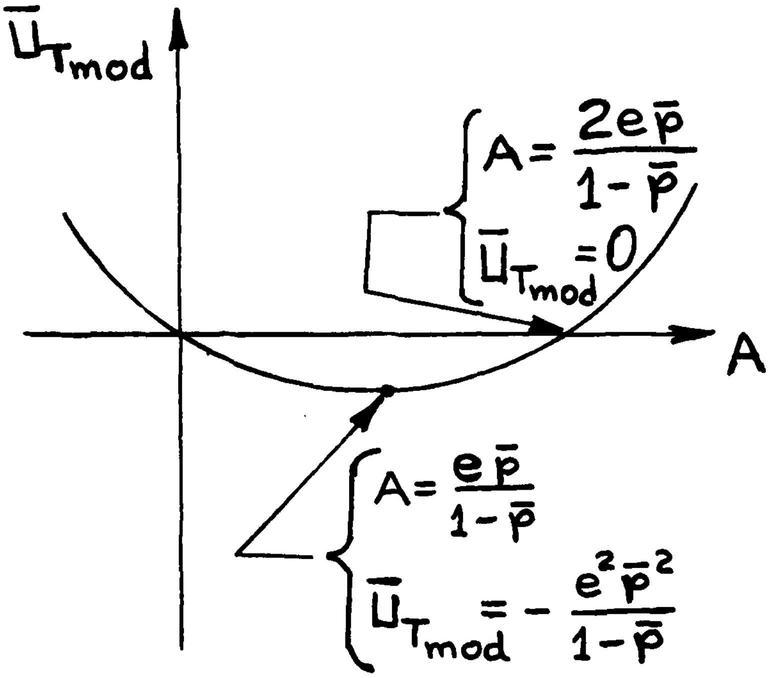


Fig. 5.6 Total Potential for a Suddenly-Loaded Half-Sine Column.

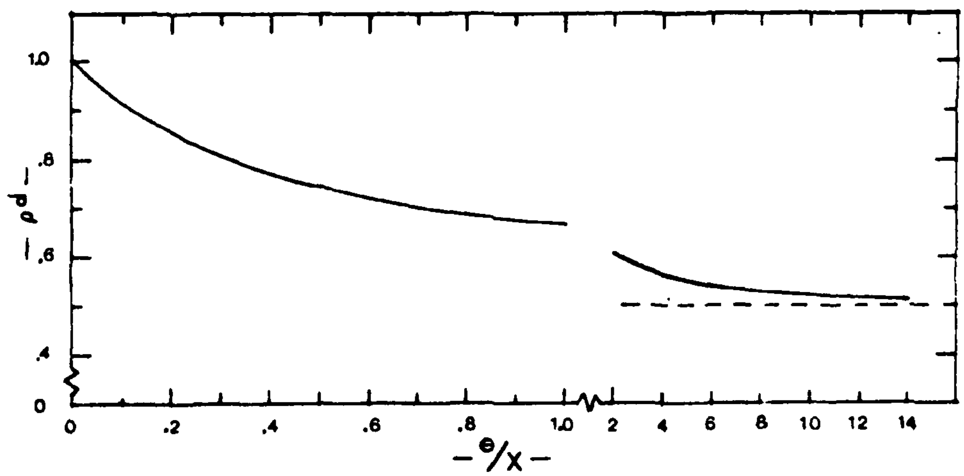


Fig. 5.7 Ratio of Dynamic to Static Load versus ratio of imperfection parameter to maximum allowable displacement (for the imperfect column).

On the basis of the above, the dynamic magnification factor, for this case, is

$$D = A_{d_{max}} / A_{st_{max}} = 2 \qquad (5.75)$$

regardless of the value of e.

ACKNOWLEDGEMENT

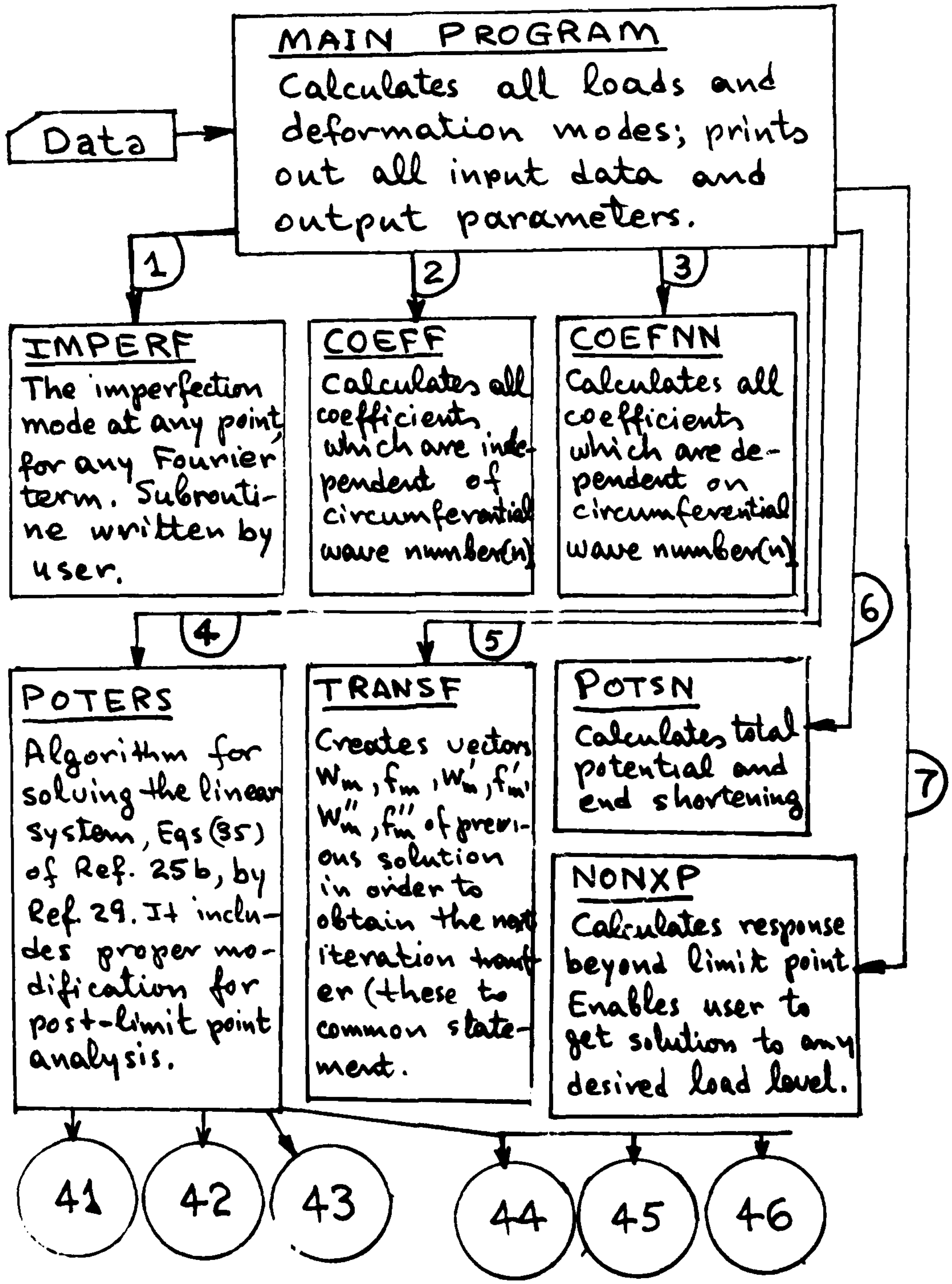
The authors wich to thank the U.S. Air Force Aeronautical Systems Division (AFSC), Patterson Air Force Base, for financially supporting the reported research. Moreover, they wish to thank Dr. V. B. Venkayya for his encouragement and support during the course of the investigation. Furthermore, the authors are grateful to Mr. Andreas Vlahinos for performing some of the calculation and for the artwork. Finally, the authors wish to thank Mrs. Nancy Starkey and Mrs. Betty Mitchell for typing the manuscript and particularly for showing patience and understanding. Mrs. Peggy Varalla's help is also gratefully acknowledged.

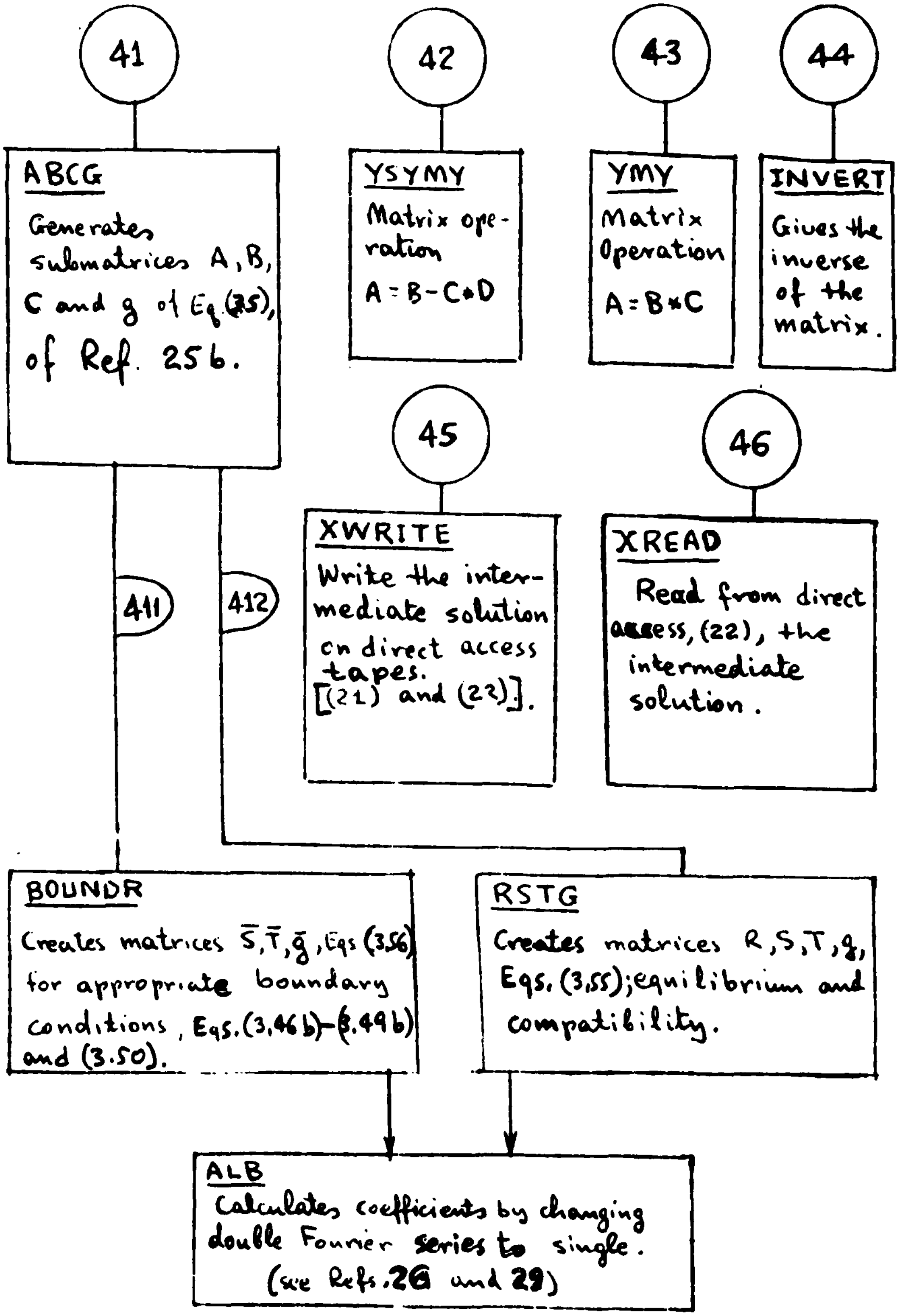
APPENDIX A

FLOW CHART

(CYLINDRICAL SHELL ANALYSIS)

I. BLOCK DIAGRAM




41
42
43
44
ABCG
Generates submatrices A, B, C and g of Eq.(3.35), of Ref. 25b.
YSYMY
Matrix operation
A = B - C*D
YMY
Matrix Operation
A = B*C
INVERT
Gives the inverse of the matrix.
45
46
411
412
XWRITE
Write the intermediate solution on direct access tapes. [(21) and (22)].
XREAD
Read from direct access, (22), the intermediate solution.
BOUNDR
Creates matrices S̄, T̄, ḡ, Eqs (3.56) for appropriate boundary conditions, Eqs.(3.46b)-(3.49b) and (3.50).
RSTG
Creates matrices R, S, T, g, Eqs. (3.55); equilibrium and compatibility.
ALB
Calculates coefficients by changing double Fourier series to single. (see Refs. 26 and 29)

COMMON CARDS

1) Common/CINTG/NEQPOT, MI (500)

NEQPOT - Number of points in axial direction

MI (500) - The order of Eq. I, MI(I) [according to Ref. 29].

2) COMMON/BOUND/LS1, LSN

Definition of boundary condition at the first point (LS1), and at the last point (LSN) of the shell.

3) Common/FIDFR/DELTA, AL1, GA1, AL2, BT2, GA2.

Coefficients of finite difference form, Δ, $\alpha^1 = -1/2\Delta$, $\gamma^1 = 1/2\Delta$, $\alpha^2 = \gamma^2 = 1/\Delta^2$, $\beta^2 = -2/\Delta^2$.

4) Common/FOURIR/KFOUR, K6, K4, K3, K2, K1.

Fourier series limit (K = KFOUR) and parameters dependent on K.

5) COMMON/GEOM/RR, DD, H_{11}, H_{12}, H_{22}, Q_{11}, Q_{12}, Q_{22}, D_{11}, D_{12}, D_{22}

Shell geometric parameters; R, D, h_{ij}, q_{ij}, d_{ij} [Eqs. (3.12) - (3.14)].

6) COMMON/FACTOR/C1, C2, ... C12.

Coefficients which are dependent on circumferential wave number.

7) COMMON/FACT2/DL1-DL4, DA1-DA4, DB2, DB3, DB4, XNI, Exxp.

Coefficients $\gamma_1, \gamma_2, \gamma_3, \gamma_4, a_1, a_2, a_3, a_4, b_2, b_3, b_4, \nu, E_{xxp}$.

8) COMMON/CDISK/I21(501),I22(501)

Direct access data set 21 and 22.

9) COMMON/FACT3/DLS,XL,XH.

parameter, XL=L, XH=t.

10) COMMON/PRES1/WM(200,5),ETM(200,5),WMP(200,5)

The vector of previous solution W_m, W_m'', and W_m' at point l for Fourier term i.

11) COMMON/PRES2/WZ(200,5),WZP(200,5),WZPP(200,5)

Imperfection mode W^o, $W^{o'}$, $W^{o''}$ at point l for Fourier term i.

12) COMMON/PRES3/FM(200,8),XFM(200,8),FMP(200,8)

The vector of previous solution f_m, f_m'', and f_m' at point l for Fourier term i.

13) COMMON/XXLOAD/XPRES

XPRES - hydrostatic pressure.

14) COMMON/NEWPT/JPS ,INXXPX

JPS - The Fourier (+1) Number for which the unknown displacement is replaced by the load factor (and treated as known).

INXXPX - See User's manual

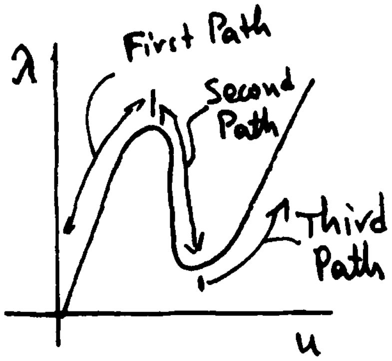
15) COMMON/SHEINN/VOUT (12,130),VPOT (7,130), IVOUT

Arrays for printout

16) COMMON/XXNNPP/XNP1, XNP3

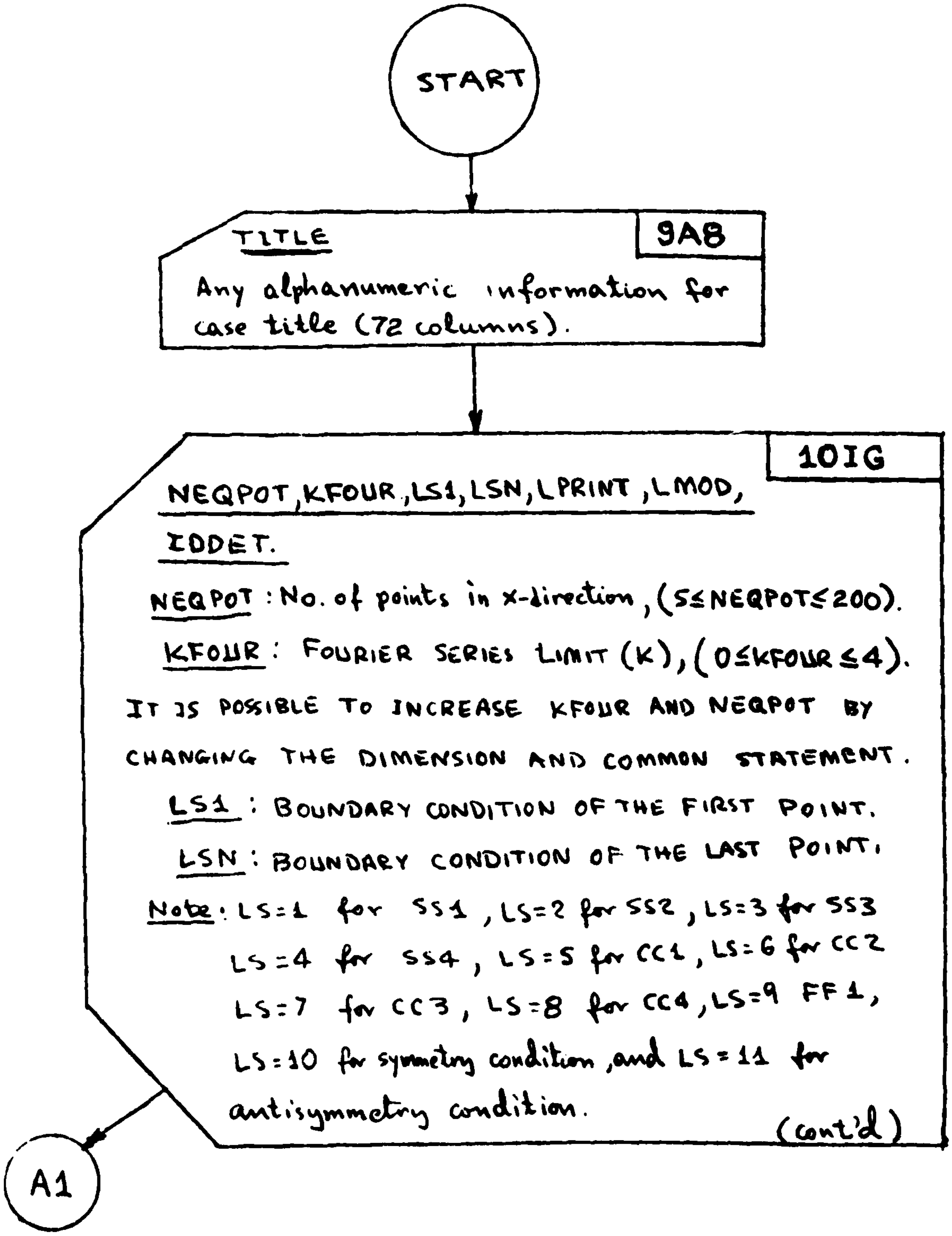
XNP1 - the limit for the first path

XNP3 - the limit for the third path.



USER'S MANUAL

II. FLOW CHART FOR DATA PREPARATION



A1

(cont'd)

LPRINT 0 = minimum printout ; 1 = maximum printout

LMOD 0 : does not print modes , 1 : prints modes

IDDET 0 : does not calculate determinant

1 : calculates determinant and prints it.

AP (12K+4,12K+4) , BP (12K+4,12K+4), CP (12K+4,12K+4)

PR (12K+4, 12K+4), GP (12K+4,1), XP(12K+4,1), T1 (12K+4), C(12K+4),

MT (12K+4) , V1 ((12K+4)*(12K+4)).

COMMON/PRES1/WM (NEQPOT,K+1), ETA (NEQPOT,K+1), WMP (NEQPOT, K+1).

COMMON/PRES2/WZ (NEQPOT,K+1), WZP (NEQPOT,K+1), WZPP(NEQPOT,K+1).

COMMON/PRES3/FM (NEQPOT,2K), XFM (NEQPOT,2K), FMP (NEQPOT,2K).

6E12.4

RR , XL, XH, ELAS, XNI

RR : Radius of the cylinder

XL : Length of the cylinder

XH : Thickness of the cylinder

ELAS : Modulus of Elasticity

XNI : Poisson's ratio.

A2

A2

6E12.4

XLAMD, YLAMD, EX, EY, RHOX, RHOY

XLAMD = $\lambda_{xx} = (1-\nu^2)A_x/t\ell_x$

YLAMD = $\lambda_{yy} = (1-\nu^2)A_y/t\ell_y$

EX : Stringer eccentricity parameter (positive inward)

EY : Ring eccentricity parameter (positive inward)

RHOX = $\rho_{xx} = EI_{x_c}/D\ell_x$

RHOY = $\rho_{yy} = EI_{y_c}/D\ell_y$

The user must write subroutine IMPERF for the definition of the imperfection and the derivatives.

WZ (I,J) = W° (positive inward)

WZP(I,J) = $W^{\circ\prime}$

WZPP(I,J) = $W^{\circ\prime\prime}$

I : 1, NEQPOT mesh point

J : 1, KFOUR+1 for all Fourier terms

Note that since $W^{\circ} = \sum_{i=0}^{K} W_i^{\circ}(x)\cos\frac{iny}{R}$

then

J = 1 for i = 0, and J = KFOUR+1 for i = KFOUR

A3

A3

INXXPX, LNXXPX, JPS, LP, RW

INXXPX = 1 for $\bar{N}_{xx}$ and p as known

INXXPX = 2 for $\bar{N}_{xx}$ as unknown instead of W(LP,JPS) (the initial solution $\bar{N}_{xx}^{m} = 0$).

INXXPX = 3 for $\bar{N}_{xx}$ as unknown instead of W(LP,JPS). (the initial soln for $\bar{N}_{xx}$ is the previous one).

INXXPX = 4 for p as unknown instead of W(LP,JPS).

LNXXPX = 1 (used for INXXPX ≠ 1) $\bar{N}_{xx}$ or p are unknown from the second step. The first step is for getting an initial solution (for the second step).

LNXXPX = 2 (used for INXXPX ≠ 1) $\bar{N}_{xx}$ and p are taken as unknowns, one step before limit pt.

JPS The Fourier number(+1) on which W(LP,JPS) is replaced by load as known parameter.

LP The mesh point on which W(LP,JPS) is replaced by load as known parameter.

RW (for INXXPX ≠ 1) The increment of W(LP,JPS). DW(LP,JPS) = RW * W(LP,JPS), where W(LP,JPS) is the last solution before it is replaced by the load as known (parameter). For the next step, the known displacement will be

$$W^{m+1}(LP,JPS) = W^{m}(LP,JPS) + DW(LP,JPS).$$

A30

A30

cont'd

After running a few examples, the user can easily select an effective RW. Figs. a and b suggest values for RW, depending on the shell behavior.

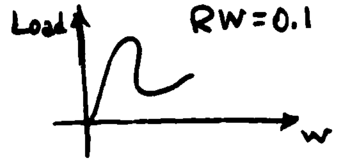


Fig. a

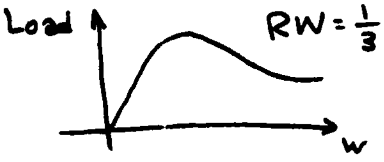
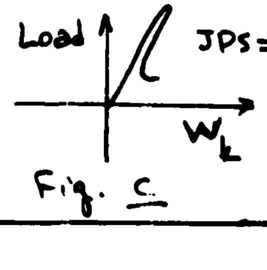


Fig. b

In order to save computer time, run examples by taking the load as known up to the limit pt. (LNXXPX = 2).

Note that LP = no. of mesh points, and JPS = Fourier No. + 1 on which W(LP, JPS) is treated as a known parameter.

If LP and JPS are set to zero, the program finds automatically their proper value by identifying the most dominant displ. parameter. Sometimes the solution does not converge because the chosen dominant parameter behaves as the ones shown on Fig. c. In this case, the user should make another choice (one for which the behavior in the post limit pt. range is similar to that of Fig. d). The user can given only one of (LP, JPS) or both, or none (by setting them to zero).



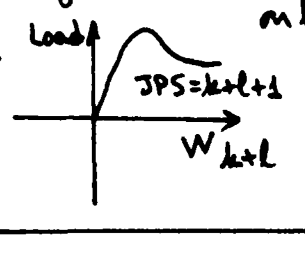


Fig. c

A31

A31

6E12.4

DLND, XNXX, XPRESS, DNXP, ACCUR, RI, XNP1, XNP3.

DLND = 1 For fixed lateral pressure p find $\bar{N}_{xx\,cr}$

DLND = 2 For fixed axial compression $\bar{N}_{xx}$, find p_{cr}

DLND = 3 Axial load and pressure are related by the factor XNXX ($\bar{N}_{xx}$=XNXX*XPRES).

XNXX For DLND=1, is the initial axial load
For DLND=2, is the fixed axial load
For DLND=3, is the factor that relates $\bar{N}_{xx}$ and p (positive XNXX means compression)

XPRES For DLND=1, is the fixed pressure
For DLND=2,3, is the initial press. (+ inward)

DNXP For DLND=1, is the increment in axial load
For DLND=2,3, is the increment in pressure.

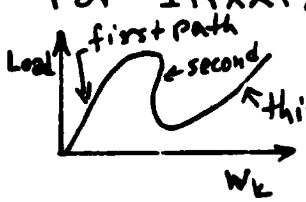
ACCUR The required load accuracy in percent.

RI Maximum number of load points

N_{xx}=XNXX + RI * DNXP (for DLND = 1)

p = XPRESS + RI * DNXP (for DLND = 2,3).

For INXXPX = 2,3,4 ; $\overset{m+1}{W}$(LP,JPS) = $\overset{m}{W}$(LP,JPS) + DW(LP,JPS).



XNP1 : The upper limit of first path

XNP3 : The upper limit of third path

The program stops automatically if XNP1 and XNP3 have been exceeded.

A4

A4

INON, INON1, INON2, INON3 — 10I6

INON = 1 The initial solution is the linear solution

INON = 2 The initial solution (at XNXX + DNXP or XPRES + DNXP) is the solution at previous step (XNXX or XPRES).

INON 1 Same as INON but for the path on which the load is unkown.

(It is recommended to take INON = 1 & INON1 = 2).

INON2 = 2 : The results (values) of WM, ETM, WMP, FM, XFM, and FMP for the last step (last run) are put on tape (16) and on permanent file. These values may be used as an initial solution for the next (restart) run of same case.

INON3 = 2 The initial solution is given. (otherwise the program should start from point just before the limit point). In this case, INON2 = 2 (for the previous run).

Note that INON2 & INON3 exist for LNXXPX = 1 and INXXPX ≠ 1.

A5

A5

1016

NNN, LNNN, ILNW

NNN – The circumferential wave number, n. The program finds the limit pt for this n.

LNNN=0 : The program does not check which n gives minimum potential energy.

LNNN=1 : The program finds the limit point for NNN and calculates the total potential for values surrounding NNN at load levels lower than the (NNN) limit point.

LNNN=2 : The program calculates the total potential only for the given load XNXX. In this case, set XNXX close to estimated limit pt.

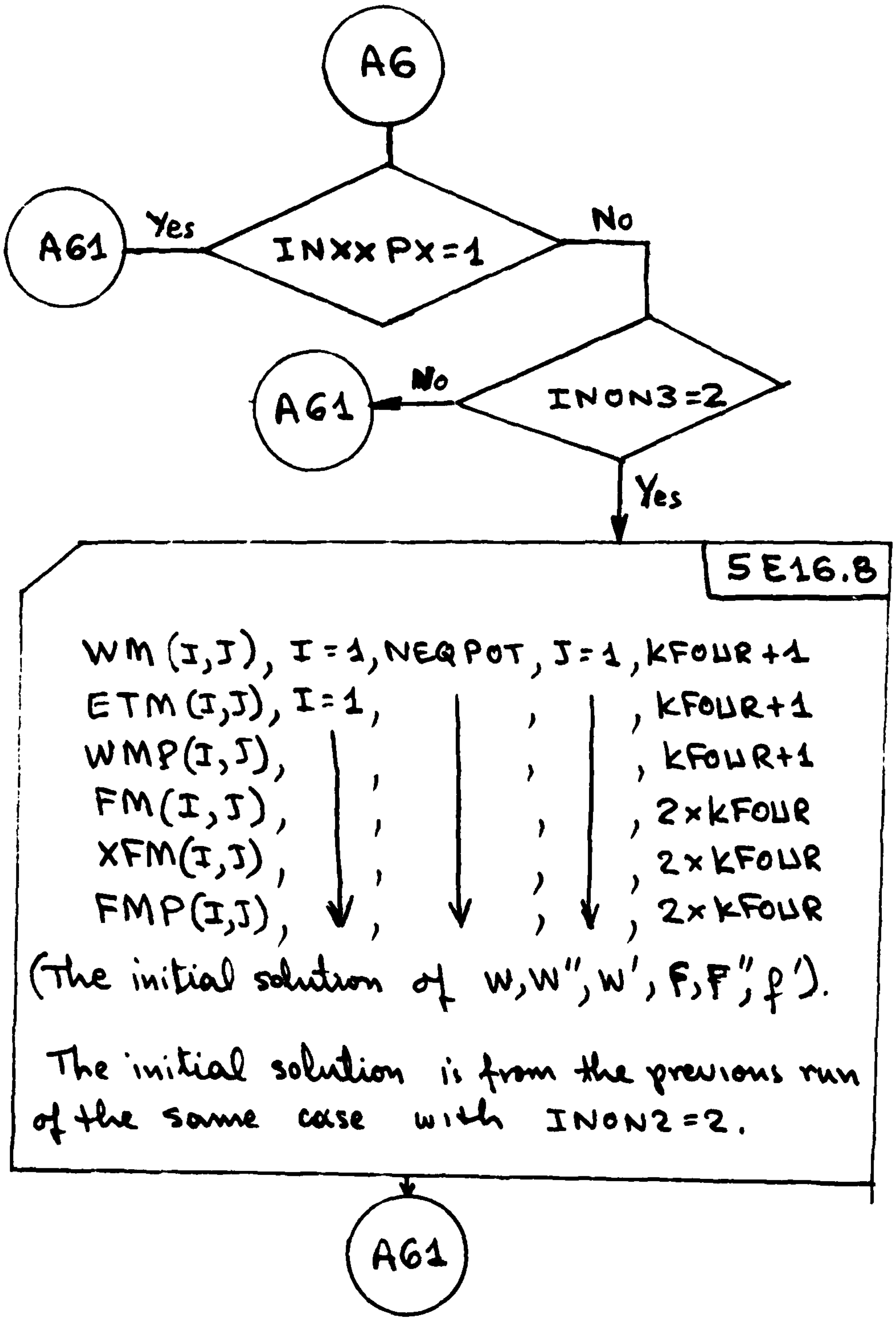
ILNW ≤ 10 : Ten is the maximum number of n-values for which the program calculates the total potential, in order to find the minimum and minimizing <u>n</u>.

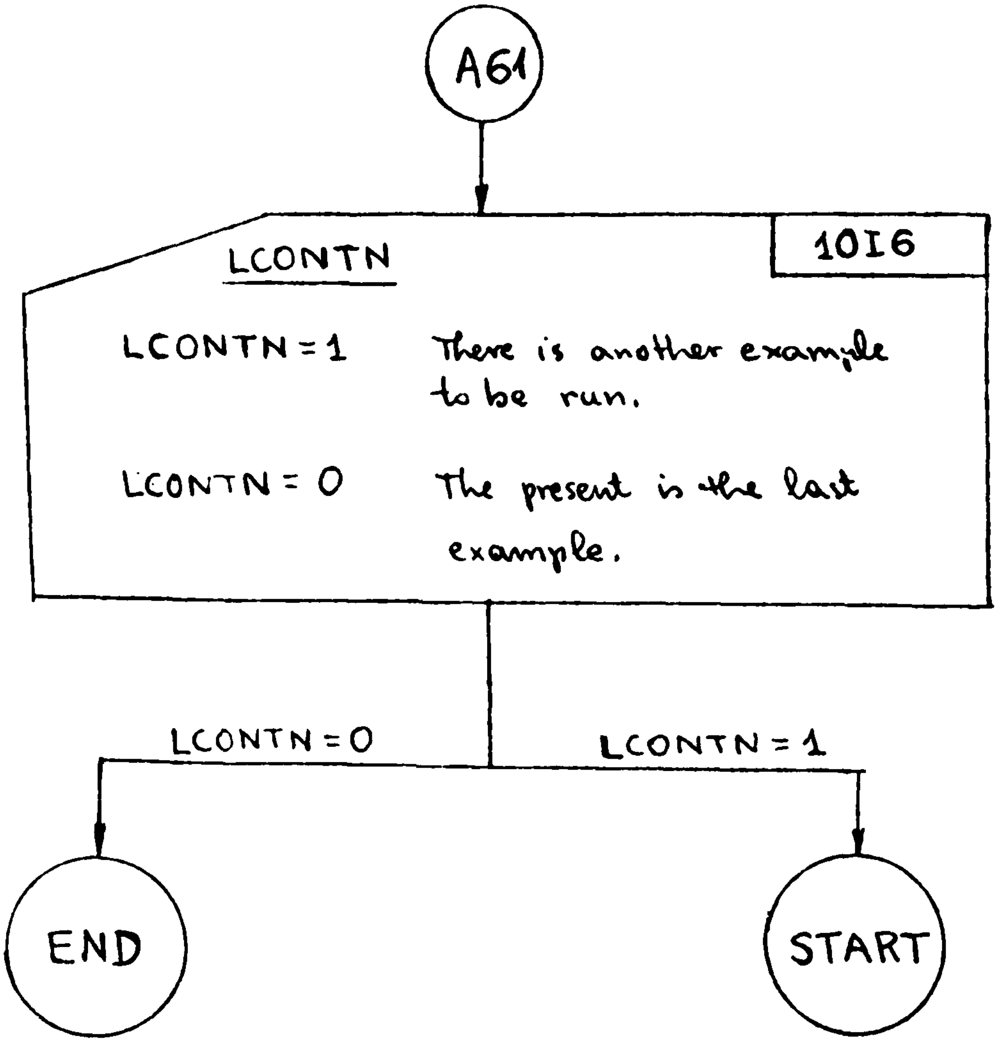
For example, if $n=5$ and ILNW=2 the program calculates the total potential for $n=5$ & 6.

If $U_T(n=6) < U_T(n=5)$ then it calculates $U_T(n=7)$; if not it calculates $U_T(n=4)$

<u>Note that</u> for INXXPX ≠ 1 LNNN should be <u>0</u>

A6


A6
A61
Yes
INXXPX=1
No
A61
No
INON3=2
Yes
5E16.8
WM (I,J), I=1,NEQPOT, J=1, KFOUR+1
ETM (I,J), I=1, , , KFOUR+1
WMP(I,J), , , , KFOUR+1
FM(I,J), , , , 2×KFOUR
XFM(I,J), , , , 2×KFOUR
FMP(I,J), , , , 2×KFOUR
(The initial solution of W, W″, W′, F, F″, f′).
The initial solution is from the previous run of the same case with INON2=2.
A61


A61
LCONTN
1016
LCONTN = 1
There is another example to be run.
LCONTN = 0
The present is the last example.
LCONTN = 0
LCONTN = 1
END
START

APPENDIX B

COMPUTER PROGRAM

(CYLINDRICAL SHELL ANALYSIS)

```
/JOB
/NOSEQ
NIRIT,CM160000,T1500.
USER.
FTN,OPT=2.
LGO,INPUT,OUTPUT,DFIL,PL=99999.
REPLACE,DFIL.
/EOR
       PROGRAM MAIN(INPUT,OUTPUT,DFIL,TAPE5=INPUT,TAPE6=OUTPUT,
      1TAPE16=DFIL,TAPE20,TAPE21,TAPE22,TAPE23)
C     POST BUCKLING OF STIFFENED CYLINDRICAL SHELLS UNDER UNIFORM AXIAL
C    COMPRESSION (NONLINEAR THEORY)
C    AN EXTENSION OF THE PROGRAM FOR LOAD LEVEL OVER THE LIMIT
C    POINT HAS BEEN DONE ON NOVEMBER 1980 IN GEORGIA TECH BY
C    SHEINMAN
       COMMON/XXLOAD/XPRES
       COMMON/CINTG/NEQPCT,MI(500)
       COMMON/BOUND/LS1,LSN
       COMMON/FIDFR/DELTA,AL1,GA1,AL2,BT2,GA2
       COMMON/FOURIE/KFOUR,K0,K4,K3,K2,K1
       COMMON/GEOM/RR,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
       COMMON/FACTOR/C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12
       COMMON/FACT2/DL1,DL2,DL3,DL4,DA1,DA2,DA3,DA4,DB2,DB3,DB4,XNI,EXXP
       COMMON/CDISK/I21(501),I22(501),I23(501)
       COMMON/FACT3/DL5,XL,XH
       COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
       COMMON/PRES2/WZ(100,5),WZF(100,5),WZPP(100,5)
       COMMON/PRES3/FM(100,8),XFM(100,8),FMP(100,8)
        COMMON/NEWPT/JPS,INXXPX
          COMMON/SHEINN/VOUT(12,130),VPOT(7,130),IVCOT
            COMMON/XXNNPP/XNP1,XNP3
           COMMON/RVKA/XIMPS
       DIMENSION WWM(5),FFM(8)
       DIMENSION TI(10)
       DIMENSION WF(2,5),XWF(2,5),FF(2,8),XFF(2,8)
       DIMENSION AP(52,52),BP(52,52),CP(52,52),FR(52,52),GP(52,1)
       DIMENSION XP(52,1),T1(52),CC(52),MT(52),V1(2704)
        DIMENSION DP(52,1)
C     2704=52*52
       DIMENSION WCON(20,5),FCON(20,8)
C     ALL THE CARDS WITH SIGN   **   IN COLUMNS 73,74  DEPEND ON  NUMBER
C OF POINTS AND  KFOUR
       EQUIVALENCE (AP(1,1),V1(1))
       CALL OPENMS(21,I21,501,0)
       CALL OPENMS(22,I22,501,0)
        CALL OPENMS(23,I23,501,0)
       ECONV=0.001
        ECONN=0.001
       MAXN=52
       MAX2=MAXN*MAXN
       NRHS=1
       NJ=100
       NW=5
       NF=8
         REWIND 16
C    NJ,NW,NF - FOR DIMENSION  --  NJ=MAXIMUM POINTS IN AXIAL DIRECTION
C  NW= MAXIMUM KFOUR+1   ,   NF= MAXIMUM 2*KFOUR  ,  MAXN=12*KFOUR+4
C IN ORDER TO INCREASE THE CAPABILITY OF THE PROGRAM FOR MANY POINTS IN
C  DIRECTION AND HIGHER LIMIT OF FOURIER SERIES THE USER HAS TO CHANGE
C  ALL THE CARDS WITH THE SIGN  **  IN COLUMNS  73 AND  74
 1111 WRITE(6,20)
      READ(5,10)(TI(I),I=1,9)
      WRITE(6,60)
      WRITE(6,10)(TI(I),I=1,9)
```

```
      READ(5,*   )NEQPOT,KFOUR,LS1,LSN,LPRINT,LMOD,IODET
      IF(LPRINT.EQ.1)LMOD=1
      READ(5,*   )RR,XL,XH,ELAS,XNI,XIMPS
      READ(5,*   )XLAMD,YLAMD,EXX,EYY,RHOX,RHOY
        IOVER=0
        JPR=0
        IVOUT=0
      EX=-EXX
      EY=-EYY
C     *************
        CALL COEFF(EX,EY,XLAMD,YLAMD,RHOX,RHOY,ELAS)
C   **************
      WRITE(6,300)NEQPOT,KFOUR,LS1,LSN
      WRITE(6,400)RR,XL,XH,ELAS,XNI,DC,EXXP
      WRITE(6,573)XLAMD,YLAMD,EXX,EYY,RHOX,RHOY
C    *********
      CALL IMPERF
C    **************
      WRITE(6,508)
      DO 85 IK=1,K1
      LK=IK-1
      WRITE(6,510)LK
      WRITE(6,520)
      XX=0.
      DO 85 I1=1,NEQPOT
      WRITE(6,509)I1,XX,WZ(I1,IK),WZP(I1,IK),WZPP(I1,IK)
      XX=XX+DELTA
   85 CONTINUE
      IF (LPRINT.NE.1) GO TO 39
      WRITE(6,500)DELTA,AL1,GA1,AL2,BT2,GA2
      WRITE(6,501)H11,H12,H22,Q11,Q12,Q22
      WRITE(6,502)D11,D12,D22,D62,DB3,DB4
      WRITE(6,503)DL1,DL2,DL3,DL4,DL5
      WRITE(6,504)DA1,DA2,DA3,DA4
   39 CONTINUE
      DO 63 I1=1,NEQPOT
      DO 64 J1=1,K1
      WM(I1,J1)=0.
      ETM(I1,J1)=0.
      WMP(I1,J1)=0.
   64 CONTINUE
      DO 65 J1=1,K2
      FM(I1,J1)=0.
      XFM(I1,J1)=0.
      FMP(I1,J1)=0.
   65 CONTINUE
   63 CONTINUE
       LP=0
       INXXPX=1
       READ(5,* )INXXP,LNXXPX,JFR,LFR,RW
       RW1=100.*RW
       WRITE(6,706) INXXP,LNXXPX,RW1
  706  FORMAT(//,2X,"INXXPX=",I5,5X,"LNXXPX=",I5/,2X,"THE INCREMENT OF
     1 W IS ",E12.4,2X,"PERCENT"/)
      READ(5,*   )OLNO,XNXX,XPREE,ONXX,ACCUR,RII,XNF1,XNP3
      IRR=RII
      IF(IRR.EQ.0)IRR=1
      IOLNO=OLNO
      ONX=ONXX
      XPRES=XPREE
      XFNX=XNXX
C XFNX=AXIAL COMPRESSION, XPRES=HYDROSTATIC PRESSURE, XNX=EITHER XFNX
C OR XPRES ACCORDING TO IOLNO
      GO TO(71,72,73),IOLNO
```

```
 71 WRITE(6,511)XFRES,XFNX,DNX,ACCUR
    XNX=XNXX
    XN11=XNXX
    GO TO 74
 72 WRITE(6,512)XFNX,XPRES,DNX,ACCUR
    XNX=XPREE
    XN11=XFRES
    GO TO 74
 73 WRITE(6,513)XNXX,XPRES,DNX,ACCUR
     IF(INXXP.EQ.1)GO TO 202
     WRITE(6,203) INXXP,IDLNO
203  FORMAT(//,2X,"THE OPTION OF INXXPX =",I5,2X,"AND IOLNO="
   1,I5,2X,"IS NOT AVAILABLE YET"/)
     STOP
202  CONTINUE
    TLMOX=XNXX
    XFNX=TLMOX*XPRES
    XNX=XPREE
    XN11=XFRES
 74 TXNX=1000.*XNX
    READ(5,*  )INON,INON1,INON2,INON3
    READ(5,*  )NNN,LNNN,ILNW
    NWAVE=NNN
      IF(INXXP.NE.1)LNNN=0
C  *********************
    CALL COEFNN(NWAVE)
C  *********************
    WRITE(6,505)NNN,INON,INON1,INON2,INON3
    IF(LPRINT.NE.1) GO TO 49
    WRITE(6,506)C1,C2,C3,C4,C5,C6
    WRITE(6,507)C7,C8,C9,C10,C11,C12
 49 CONTINUE
    ILR=0
    LICON=1
    IPOTT=0
    CALL SECOND(TIM1)
    WRITE(6,733)TIM1
    TIM2=TIM1
    TIM4=TIM2
    IINN=0
      IF(INON2.EQ.2)WRITE(16,10)(TI(I1),I1=1,9)
      IF(INXXP.NE.1.AND.INON3.EQ.2) GO TO 213
      GO TO 555
213   READ(5,214) ((WM(I1,J1),I1=1,NEQPOT),J1=1,K1)
      READ(5,214) ((LTM(I1,J1),I1=1,NEQPOT),J1=1,K1)
      READ(5,214) ((W1F(I1,J1),I1=1,NEQPOT),J1=1,K1)
      READ(5,214) ((FM(I1,J1),I1=1,NEQPOT),J1=1,K2)
      READ(5,214) ((XFM(I1,J1),I1=1,NEQPOT),J1=1,K2)
      READ(5,214) ((FMP(I1,J1),I1=1,NEQPOT),J1=1,K2)
214   FORMAT(5E16.8)
      GO TO 815
555 LN=1
    IF(IOLNO.EQ.1)XFNX=XNX
    IF(IOLNO.EQ.2.OR.IOLNO.EQ.3)XPRES=XNX
    IF(IOLNO.EQ.3)XFNX=TLMOX*XPRES
    IDET=IDDET
    CALL POTERS(IDET,NRHS,MAXN,AP,BP,CP,GP,PR,XP,CC,MT,T1,V1,MAX2,
   1IXPM,DETM,XFNX,LN,NJ,NW,NF,LP,DF)
    IF(LPRINT.NE.1) GO TO 101
    CALL SECOND(TIM3)
    TIM1=TIM3-TIM2
    TIM2=TIM3
    TIM4=TIM3
    WRITE(6,501)NWAVE,XFNX,XPRES,TIM1
```

```
  101 CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,1,LPRINT,WW,XNP,LP)
  444 IDET=IODET
      IF(IDLND.EQ.1)XFNX=XNX
      IF(IDLND.EQ.2.OR.IDLND.EQ.3)XPRES=XNX
      IF(IDLND.EQ.3)XFNX=TLMOX*XPRES
      IMAX=1
      WMAX=0.
      ITER=0
      DO 102 J1=1,K1
  102 WMAX=WMAX+WM(1,J1)
      DO 103 I1=2,NEQPOT
      WMM=0.
      DO 104 J1=1,K1
  104 WMM=WMM+WM(I1,J1)
      IF(ABS(WMM).LE.ABS(WMAX)) GO TO 103
      WMAX=WMM
      IMAX=I1
  103 CONTINUE
      JWMAX=1
      WWM(1)=WM(IMAX,1)
      AWWM=WWM(1)
      IF(K1.EQ.1) GO TO 1051
      DO 105 J1=2,K1
      WWM(J1)=WM(IMAX,J1)
      IF(ABS(WWM(J1)).LE.ABS(AWWM)) GO TO 105
      AWWM=WWM(J1)
      JWMAX=J1
  105 CONTINUE
 1051 JFMAX=1
      FFM(1)=FM(IMAX,1)
      AFFM=FFM(1)
      IF(K2.EQ.1) GO TO 333
      DO 106 J1=2,K2
      FFM(J1)=FM(IMAX,J1)
      IF(ABS(FFM(J1)).LE.ABS(AFFM)) GO TO 106
      AFFM=FFM(J1)
      JFMAX=J1
  106 CONTINUE
  333 LN=2
      ITER=ITER+1
      IF(ITER.LE.10) GO TO 113
      WRITE(6,114)ITER
      GO TO 9999
  113 CALL POTERS(IDET,NRHS,MAXN,AP,BP,CP,GP,FR,XP,CC,MT,T1,V1,MAX2,
     1IXPM,DETM,XFNX,LN,NJ,NW,NF,LP,DF)
      IF(LPRINT.NE.1) GO TO 111
      CALL SECOND(TIM3)
      TIM1=TIM3-TIM2
      TIM2=TIM3
      WRITE(6,112)ITER,NWAVE,XFNX,XPRES,TIM1
  111 CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,1,LPRINT,WW,XNP,LP)
      DO 115 J1=1,K1
      IF(WM(IMAX,J1).NE.0.) GO TO 57
      WCON(ITER,J1)=0.
      GO TO 115
   57 CONTINUE
      WCON(ITER,J1)=ABS((WM(IMAX,J1)-WWM(J1))/WM(IMAX,J1))
  115 CONTINUE
      WCH=WCON(ITER,JWMAX)
      IWH=JWMAX
      DO 116 J1=1,K2
      IF(FM(IMAX,J1).NE.0.) GO TO 55
      FCON(ITER,J1)=0.
      GO TO 116
```

```
 58 CONTINUE
    FCON(ITER,J1)=ABS((FM(IMAX,J1)-FFM(J1))/FM(IMAX,J1))
116 CONTINUE
    FCH=FCON(ITER,JFMAX)
    IFH=JFMAX
    IF(LPRINT.NE.1) GO TO 117
    WRITE(6,118)ITER,WCH,FCH
    WRITE(6,119)(J1,WCON(ITER,J1),J1=1,K1)
    WRITE(6,119)(J1,FCON(ITER,J1),J1=1,K2)
117 IF(WCH.GT.ECONV) GO TO 194
    IF(FCH.GT.ECONV) GO TO 194
    GO TO 195
194 IF(ITER.LE.2) GO TO 196
    IF(WCON(ITER,IWH).GT.WCON(ITER-1,IWH)) GO TO 197
    IF(FCON(ITER,IFH).GT.FCON(ITER-1,IFH)) GO TO 197
    GO TO 196
197 IF(XNX.NE.XN11) GO TO 198
    WRITE(6,991)XNX
    GO TO 9999
196 DO 131 J1=1,K1
131 WWM(J1)=WM(IMAX,J1)
    DO 132 J1=1,K2
132 FFM(J1)=FM(IMAX,J1)
    GO TO 333
195 IF(IDLND.EQ.1)XFNX=XNX
    IF(IDLND.EQ.2.OR.IDLND.EQ.3)XPRES=XNX
    IF(IDLND.EQ.3)XFNX=TLMDX*XPRES
      IVOUT=IVOUT+1
    CALL PCTSN(PCT,PCTM,STRY,STRA,1,1,1,XFNX)
    CALL SECOND(TIM3)
    TIM1=TIM3-TIM4
    TIM2=TIM3
    TIM4=TIM3
    WRITE(6,241)XFNX,XPRES,NWAVE,ITER,TIM1
    WRITE(6,242)PCT,PCTM,STRY,STRA
    IF(IDET.EQ.1) WRITE(6,243)DETM,IXPM
    IF(LMOD.NE.1) GO TO 476
    CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,2,3,AW,XNP,LP)
476 CONTINUE
    IF(LNNN.EQ.2.AND.LICON.NE.10) GO TO 566
    IF(LICON.NE.10) GO TO 629
    ILR=1
    GO TO 777
629 XNX1=XNX
    IINN=IINN+1
      VOUT(1,IVOUT)=XFNX
      VOUT(2,IVOUT)=XPRES
      VOUT(3,IVOUT)=PCT
      VOUT(4,IVOUT)=PCTM
      VOUT(5,IVOUT)=STRY
      VOUT(6,IVOUT)=STRA
      VOUT(7,IVOUT)=0.
      VOUT(9,IVOUT)=0.
      IF(LP.EQ.0)GO TO 7311
      VOUT(7,IVOUT)=WM(LP,1)
      VOUT(9,IVOUT)=WM(LP,2)
7311  NSHR=(NEQPOT+1)/2
      VOUT(8,IVOUT)=WM(NSHR,1)
      VOUT(10,IVOUT)=WM(NSHR,2)
      VOUT(11,IVOUT)=ITER
      VOUT(12,IVOUT)=NWAVE
     IF(INXXP.NE.1.AND.LNXXPX.EQ.1)GO TO 615
     IF(INXXP.NE.1.AND.LNXXPX.EQ.2.AND.IOVER.EQ.1)GO TO 615
    IF(IINN.LE.IRR)GO TO 721
```

```
      WRITE(6,722)IINN
      GO TO 9999
  721 CONTINUE
      XNX=XNX+DNX
      IF(TXNX.GT.XNX) GO TO 244
      DNX=DNX/2.
      XNX=XNX-DNX
      ADN=DNX*100./XNX
      IINN=IINN-1
      IF(ADN.GT.ACCUR) GO TO 244
      XNX=XNX1
      GO TO 819
  244 IF(INON.EQ.1) GO TO 555
      REWIND 20
      WRITE(20)((WM(I1,J1),J1=1,K1),I1=1,NEQPOT),((ETM(I1,J1),J1=1,K1)
     1,I1=1,NEQPOT),((WMP(I1,J1),J1=1,K1),I1=1,NEQPOT),((FM(I1,J1)
     2,J1=1,K2),I1=1,NEQPOT),((XFM(I1,J1),J1=1,K2),I1=1,NEQPOT),
     3((FMP(I1,J1),J1=1,K2),I1=1,NEQPOT)
      GO TO 444
  198 IF(LICON.NE.10) GO TO 429
      ILR=0
      GO TO 777
  429 IF(LNNN.EQ.0) GO TO 249
      IPOTT=IPOTT+1
      IF(IPOTT.GT.1) GO TO 249
      POTT=POT
      PXNX=XNX-DNX
  249 ADN=DNX*100./XNX
      WRITE(6,545)NWAVE,XNX
        IOVER=1
      TXNX=XNX
      XNX=XNX-DNX
      IINN=0
      IF(ADN.LE.ACCUR) GO TO 819
      DNX=DNX/2.
      XNX=XNX-DNX
      IF(INON.EQ.1) GO TO 555
      REWIND 20
      READ(20)  ((WM(I1,J1),J1=1,K1),I1=1,NEQPOT),((ETM(I1,J1),J1=1,K1)
     1,I1=1,NEQPOT),((WMP(I1,J1),J1=1,K1),I1=1,NEQPOT),((FM(I1,J1)
     2,J1=1,K2),I1=1,NEQPOT),((XFM(I1,J1),J1=1,K2),I1=1,NEQPOT),
     2((FMP(I1,J1),J1=1,K2),I1=1,NEQPOT)
      GO TO 444
  819 IF(IDLND.EQ.1)  XFNX=XNX
      IF(IDLND.EQ.2.OR.IDLND.EQ.3)XPRES=XNX
      IF(IDLND.EQ.3)XFNX=TLMDX*XPRES
      WRITE(6,785)NWAVE,XFNX,XPRES,POT,POTM,STRY,STRA
 815    IF(INXXP.EQ.1)GO TO 816
        INXXPX=INXXP
        WW=0.
        IF(LPR.EQ.0)GO TO 944
        LP=LPR
        IF(JPR.EQ.0)GO TO 945
        JPS=JPR
        GO TO 946
  945   DO 947 J1=1,K1
        IF(ABS(WW).GE.ABS(WM(LP,J1)))GO TO 947
        WW=WM(LP,J1)
        JPS=J1
  947   CONTINUE
        GO TO 946
  944   IF(JPR.EQ.0)GO TO 941
        JPS=JPR
        DO 942 I1=1,NEQPOT
```

```
      IF(ABS(WW).GE.ABS(WM(I1,JPS)))GO TO 942
      WW=WM(I1,JPS)
      LP=I1
942   CONTINUE
      GO TO 943
941   CONTINUE
      DO 921 I1=1,NEQPOT
      DO 921 J1=1,K1
      IF(ABS(WW).GE.ABS(WM(I1,J1)))GO TO 921
      WW=WM(I1,J1)
      LP=I1
      JPS=J1
921   CONTINUE
943   CONTINUE
      IF(LP.LE.2)LP=3
      IF(LP.GE.NEQPOT-1)LP=NEQPOT-2
946   WW=WM(LP,JPS)
      DWW=RW*WW
      CALL NONXP(XFNX,WW,DWW,LP,INON1,IDDET,NRHS,MAXN,AP,BP,CP,
     1 DP,GP,PR,XP,CC,MT,T1,V1,MAX2,NJ,NW,NF,LPRINT,WF,XWF,FF,XFF,
     2 ECONV,ECONN,NWAVE,LMOD,IRR,INON2,INON3)
      GO TO 9999
816   CONTINUE
C  CALCULATION OF CRITICAL WAVE NUMBER
566   WRITE(6,2.)
      IF(LNNN.EQ.0) GO TO 9999
      WRITE(6,584)
      ILR=1
      NWPRIN=NWAVE
      NWAVE=NWAVE+1
      IF(LNNN.EQ.2)  POTT=POT
      POTMIN=POTT
      NMIN=NNN
      INWAVE=0
      ISTOP=0
      I9=0
      POT=POTT
777   IF(IDLND.EQ.1)XFNX=XNX
      IF(IDLND.EQ.2.OR.IDLND.EQ.3)XPRES=XNX
      IF(IDLND.EQ.3)XFNX=TLMDX*XPRES
      IF(ILR.EQ.1) GO TO 778
      WRITE(6,582)NWAVE,XFNX,XPRES
      GO TO 9999
778   INWAVE=INWAVE+1
      IINN=0
      IF(LNNN.EQ.2) FXNX=XNX
      XNX=FXNX
      LICON=10
      IF(INWAVE.EQ.1)     GO TO 391
      NWPRIN=NWAVE
      IF(I9.GE.1) GO TO 694
      IF(POTMIN.LE.POT) GO TO 139
      POTMIN=POT
      NMIN=NWAVE
      NWAVE=NWAVE+1
      GO TO 391
139   IF(NWAVE.LE.NNN+1) GO TO 549
      ISTOP=1
      GO TO 185
549   I9=I9+1
      IF(I9.GT.1) GO TO 694
      NWAVE=NNN-1
      GO TO 391
694   IF(POTMIN.LE.POT) GO TO 695
```

```
      PCTMIN=POT
      NMIN=NWAVE
      NWAVE=NWAVE-1
      GO TO 391
 695  ISTOP=1
      GO TO 185
 391  CALL COEFNN(NWAVE)
      CALL SECOND(TIM2)
 185  WRITE(6,581)NWPRIN,PXNX,POT,TIM2
      IF(NWAVE.LE.0) GO TO 9999
      IF(ISTOP.NE.1) GO TO 798
      WRITE(6,789)XNX,PCTMIN,NMIN
      GO TO 9999
 798  IF(INWAVE.GT.ILNW) GO TO 9999
      GO TO 555
9999  READ(5,*   )LCONTN
         WRITE(6,1021)
1021     FORMAT(//,2X,"RESULTS FOR CONCLUSIONS"//,2X,
     1"RRRRRRRRRRRRRRRRRRRRRRRRRRRR"///,2X,"AXIAL LOAD",2X,
     2"PRESSURE",3X,"POTENTIAL",3X,"MODIF.  POT",2X,
     3"END-SHORTY=0",2X,"AVE ENDS",3X,"W(LP,0)",3X,"W(MID,0)",
     46X,"W(LP,1)",3X,"W(MID,1)",3X,"ITER  NWAVE"/)
         WRITE(6,1022)((VOUT(I1,J1),I1=1,12),J1=1,IVOUT)
1022     FORMAT(10E12.4,2F4.1)
         WRITE(6,1023)
1023     FORMAT(////,2X,"MODIFICATION OF POTENTIAL ENERGY"///,
     13X,"POTENTIAL",4X,"MOD. POTENTIAL",5X,"PE1-F0",9X,
     2"PE2-FI",9X,"PE3-WI",8X,"PE4-NXXW",5X,"PE5-PW",6X,
     3"PE6-NXX",6X,"PE7-A2NXX"/)
         WRITE(6,1024)(VOUT(3,J1),VOUT(4,J1),(VPOT(I1,J1),I1=
     11,7),J1=1,IVOUT)
1024     FORMAT(9E14.6)
      IF(LCONTN.EQ.1) GO TO 1111
  20  FORMAT(1H1)
  10  FORMAT(1H0,9A8)
  60  FORMAT(//25H BEGINNING OF NEXT CASE      ///)
 100  FORMAT(10I6)
 200  FORMAT(6E12.4)
 300  FORMAT(//,2X,"NO. OF POINTS=",I8,2X,"KFOUR=",I8,2X,"BOUND.CON OF
     1POINT 1=",I8,2X,"BOUND.CON OF POINT NEQPOT=",I8)
 400  FORMAT(//,2X,"R=",E12.4,2X,"XL=",E12.4,2X,"XH=",E12.4,2X,
     1"ELAS=",E12.4,2X,"XNI=",E12.4,2X,"D0=",E12.4,2X,"EXXP=",E12.4)
 508  FORMAT(//,2X,"THE IMPERFECTION FORM IN AXIAL DIRECTION IS"/)
 510  FORMAT(//,2X,"THE IMPERFECTION FOR CIRCUMFERENTIAL WAVE  ",I6/)
 520  FORMAT(//,4X,"POINT",9X,"LENGTH",12X,"WZ",14X,"WZP",13X,"WZPP"/)
 509  FORMAT(I10,4E16.6)
 500  FORMAT(//,2X,"DELTA=",E12.4,2X,"AL1=",E12.4,2X,"GA1=",E12.4,2X,
     1"AL2=",E12.4,2X,"BT2=",E12.4,2X,"GA2=",E12.4)
 501  FORMAT(//,2X,"H11=",E12.4,2X,"H12=",E12.4,2X,"H22=",E12.4,2X,
     1"Q11=",E12.4,2X,"Q12=",E12.4,2X,"Q22=",E12.4)
 502  FORMAT(//,2X,"D11=",E12.4,2X,"D12=",E12.4,2X,"D22=",E12.4,2X,
     1"DB2=",E12.4,2X,"DB3=",E12.4,2X,"DB4=",E12.4)
 503  FORMAT(//,2X,"DL1=",E12.4,2X,"DL2=",E12.4,2X,"DL3=",E12.4,2X,
     1"DL4=",E12.4,2X,"DL5=",E12.4)
 504  FORMAT(//,2X,"DA1=",E12.4,2X,"DA2=",E12.4,2X,"DA3=",E12.4,2X,
     1"DA4=",E12.4)
 511  FORMAT(//,2X,"FOR FIXED PRESSURE OF ",E12.4,2X,"THE INITIAL AXIAL
     1LOAD IS ",E12.4,2X,"THE INCREMENT OF AXIAL LOAD IS ",E12.4/
     12X,"AND THE ACCURACY (PERCENT) OF THE AXIAL LOAD IS ",E12.4)
 512  FORMAT(//,2X,"FOR FIXED AXIAL LOAD OF ",E12.4,2X,"THE INITIAL HYDR
     1OSTATIC PRESSURE IS",E12.4/2X,"THE INCREMENT OF THE HYDROSTATIC PR
     2ESSURE IS",E12.4,2X,"AND THE ACCURACY (PERCENT) OF THE HYDROSTATIC
     3 PRESSURE IS",E12.4)
 505  FORMAT(//,2X,"THE CIRCUMFERENTIAL WAVE NUMBER=",I4,
```

```
   12X,"INON=",I5,2X,"INON1=",I5,2X,"INON2=",I5,2X, ...
506 FORMAT(//,2X,"C1=",E12.4,2X,"C2=",E12.4,2X,"C3=",E12.4,2X,
   1"C4=",E12.4,2X,"C5=",E12.4,2X,"C6=",E12.4)
507 FORMAT(//,2X,"C7=",E12.4,2X,"C8=",E12.4,2X,"C9=",E12.4,2X,
   1"C10=",E12.4,2X,"C11=",E12.4,2X,"C12=",E12.4)
793 FORMAT(//,2X,"ELAPSED TIME=",E12.4,2X,"SECONDS"/)
201 FORMAT(//,2X,"INITIAL SOLUTION (FROM LINEAR)  FOR N=",I8,2X,"NX="
   1,E12.4,2X,"P=",E12.4/2X,"TIME COPUTATION =",E12.4,2X,"SECONDS"/
   22X,"==========================================================
   3=========================================")
114 FORMAT(//,2X,"END OF THIS CASE BECAUSE ITER GREATER THAN ",I8)
112 FORMAT(//,2X,"SOLUTION FOR ITERATION ",I8,2X,"N=",I8,2X,"NX=",
   1E12.4,2X,"P=",E12.4/2X,"TIME COMPUTATION =",E12.4,2X,"SECONDS"/
   22X,"==========================================================
   3=========================================")
118 FORMAT(//,2X,"ITER=",I8,2X,"WCH=",E15.5,2X,"FCH=",E15.5/)
991 FORMAT(//,2X,"THE SOLUTION IS NOT CONVERGED  APARENTLY THE INITIAL
   1LOAD SHOULD BE LESS THAN ",E12.4)
241 FORMAT(//,2X,"NONLINEAR SOLUTION FOR NX=",E12.4,2X,"P=",E12.4,2X,
   1"N=",I8,2X,"IS OBTAINED BY ",I8,2X,"ITERATIONS"/2X,"TIME COMPUTATI
   2ON=",E12.4,2X,"SECONDS"/2X,"OKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOK
   3OK","OKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOK")
242  FORMAT(//,2X,"POTENTIAL ENERGY=",E17.7,2X,"MODIFIED OF POTENTIAL
   1 ENERGY=",E17.7/,2X,"END SHORTENING FOR Y=0. IS ",E17.7
   2,2X,"AVERAGE END SHORTENING IS ",E17.7)
243 FORMAT(//,2X,"DETEMINANT=",E12.4,2X,"IXPM=",I10)
785 FORMAT(//,2X,"CRITICAL LOAD FOR N=",I8,2X,"IS NX=",E12.4,2X,
   1"P=",E12.4/,2X,"POTENTIAL ENERGY=",E17.7,2X,"MODIFIED OF POTENTIAL
   2 ENERGY=",E17.7/,2X,"END SHORTENING FOR Y=0.=",E15.5,
   32X,"AVERAGE END SHORTENING=",E15.5/,2X,"OKOKOKOKOKOKOKOKOKOK
   4OKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOKOK")
119 FORMAT(5(I8,E15.5))
545 FORMAT(///2X,"FOR N=",I8,2X,"THE LOAD  ",E12.4,2X,"IS OVER THE LIM
   1IT POINT")
573 FORMAT(//,2X,"XLAMD=",E12.4,2X,"YLAMD=",E12.4,2X,"EX=",E12.4,2X,
   1"EY=",E12.4,2X,"RHOX=",E12.4,2X,"RHOY=",E12.4)
582 FORMAT(//,2X,"FOR N=",I8,2X,"NX=",E12.4,2X,"P=",E12.4,2X,"THE SOLU
   1ION IS NOT CONVERGED"/2X,"PROBABLY THE CRITICAL LOAD IS SMALLER TH
   2AN THESE LOADS")
789 FORMAT(//,2X,"FOR   =",E12.4,2X,"THE MINIMUN POTENTIAL ENERGY IS "
   1,E17.7,2X,"AND THE CRITICAL WAVE NUMBER IS ",I8)
564 FORMAT(//,2X,"DETERMINATION OF THE CRITICAL CIRCUMFERENTIAL WAVE
   1 NUMBER (.)"///2X,"**********************************************
   2*********"/2X,"**********************************************
   3****************")
513 FORMAT(//,2X,"THE RELATION BETWEEN AXIAL LOAD AND THE PRESSURE IS
   1 NX=",E12.4,2X,"MULTIPLY BY THE PRESSURE",E12.4/2X,
   2"THE INCREMENT OF THE PRESSURE IS",E12.4/2X,
   3"AND THE ACCURACY (PERCENT) OF THE PRESSURE IS",E12.4)
581 FORMAT(//,2X,"WAVE NO =",I10,2X,"LOAD=",E15.5,2X,
   1"POTENTIAL ENERGY=",E17.7,2X,"ELAPSED TIME=",E12.4,2X,"SECONDS")
722 FORMAT(//,2X,"END OF THIS CASE BECAUSE  NUMBER OF LOAD POINTS IS
   1GREATER THAN",I5)
    END
     SUBROUTINE NONXP(XNXX,WW,DWW,LP,INON,IDDET,NRHS,MAXN,AP,BP
   1,CP,DP,GF,PR,XP,CC,MT,T1,V1,MAX2,NJ,NW,NF,LPRINT,WF,XWF,FF,
   2 XFF,LCONV,LCONN,NWAVL,LMOD,IRR,INON2,INON3)
    COMMON/XXLOAD/XPRES
    COMMON/CINTG/NEQPCT,MI(500)
    COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
    COMMON/PRES1/W1(100,5),ETM(100,5),WMP(100,5)
    COMMON/PRES2/W2(100,5),WZP(100,5),WZPP(100,5)
    COMMON/PRES3/FM(100,5),XFM(100,5),FMP(100,5)
     COMMON/NEWMT/JPS,INXXPX
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      COMMON/SHELL/VCUT(12,130),VPOT(7,100),IW...
        COMMON/XXNNPP/XNP1,XNP3
      DIMENSION WWM(5),FFM(8)
        DIMENSION SWM(100,5),SETM(100,5),SWMP(100,5)
        DIMENSION SFM(100,8),SXFM(100,8),SFMP(100,8)
      DIMENSION WF(2,NW),XWF(2,NW),FF(2,NF),XFF(2,NF)
      DIMENSION AP(MAXN,MAXN),BP(MAXN,MAXN),CP(MAXN,MAXN)
      DIMENSION PR(MAXN,MAXN),DP(MAXN,1),GP(MAXN,1),XP(MAXN,1)
      DIMENSION T1(MAXN),CC(MAXN),V1(MAX2)
      DIMENSION MT(MAXN)
      DIMENSION WCON(20,5),FCON(20,8),XCON(20)
      WRITE(6,100) INXXPX,XNXX,XPRES,LP,JPS,WW,DWW
100   FORMAT(//,2X,"THE SOLUTION IS SWITCH TO EITHER NXX OR P AS UNKNOW
     1N INSTEAD OF W(LP,JPS)"/,2X,
     2"ACCORDING TO INXXPX PARAMETER--(SEE USERS MANUAL)"/,2X,
     3"INXXPX=",I5,2X,"THE INITIAL AXIAL LOAD=",E12.4,2X,
     4"THE INITIAL PRESSURE LOAD=",E12.4/,2X,
     5"THE INITIAL GIVEN DISPLACEMENT AT MERIDIONAL POINT LP=",I5,2X,
     6"AND FOURIER TERM+1 JPS=",I5/,2X,"IS W=",E12.4,2X,"AND THE INCREME
     7NT OF W IS DWW=",E12.4/,2X,"NXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXNPXXNPN
     8NXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXPNXXP"/)
      CALL SECOND(TIM1)
      IINN=0
          IASF=0
          DCWW=DWW
      IF(INXXPX.EQ.4)XNP=XNXX
      IF(INXXPX.NE.4)XNP=XPRES
       XNPS=XNP
       IFATH=1
       IF(INON3.EQ.2)IPATH=2
      IF(INON.EQ.2)GO TO 444
555 LN=1
    IDET=IDDET
    CALL POTERS(IDET,NRHS,MAXN,AP,BP,CP,GP,PR,XP,CC,MT,T1,V1,MAX2,
   1IXPM,DETM,XNXX,LN,NJ,NW,NF,LP,DP)
    CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,1,LPRINT,WW,XNP,LP)
     IF(INXXPX.EQ.4)XPRES=XNP
     IF(INXXPX.NE.4)XNXX=XNP
444 IDET=IDDET
      IF(IASF.NE.0)GO TO 214
      DO 213 I1=1,NEQPOT
      DO 211 J1=1,K1
      SWM(I1,J1)=WM(I1,J1)
      SETM(I1,J1)=ETM(I1,J1)
211   SWMP(I1,J1)=WMP(I1,J1)
      DO 212 J1=1,K2
      SFM(I1,J1)=FM(I1,J1)
      SXFM(I1,J1)=XFM(I1,J1)
212   SFMP(I1,J1)=FMP(I1,J1)
213   CONTINUE
      SXNP=XNP
       SWW=WW-DWW
214    CONTINUE
     XNN=XNP
    IMAX=1
    WMAX=0.
    ITER=0
    DO 102 J1=1,K1
102 WMAX=WMAX+WM(1,J1)
    DO 103 I1=2,NEQPOT
    WMM=0.
    DO 104 J1=1,K1
104 WMM=WMM+WM(I1,J1)
    IF(ABS(WMM).LE.ABS(WMAX)) GO TO 103
```

```
      WMAX=WMM
      IMAX=I1
  103 CONTINUE
        IF(IMAX.EQ.LP.AND.IMAX.NE.NEQPOT) IMAX=IMAX+1
      JWMAX=1
      WWM(1)=WM(IMAX,1)
      AWWM=WWM(1)
      IF(K1.EQ.1) GO TO 1051
      DO 105 J1=2,K1
      WWM(J1)=WM(IMAX,J1)
      IF(ABS(WWM(J1)).LE.ABS(AWWM)) GO TO 105
      AWWM=WWM(J1)
      JWMAX=J1
  105 CONTINUE
 1051 JFMAX=1
      FFM(1)=FM(IMAX,1)
      AFFM=FFM(1)
      IF(K2.EQ.1) GO TO 333
      DO 106 J1=2,K2
      FFM(J1)=FM(IMAX,J1)
      IF(ABS(FFM(J1)).LE.ABS(AFFM)) GO TO 106
      AFFM=FFM(J1)
      JFMAX=J1
  106 CONTINUE
  333 LN=2
      ITER=ITER+1
      IF(ITER.LE.14) GO TO 113
      WRITE(6,114)ITER
      GO TO 403
  114    FORMAT(//,2X,"END OF THIS CASE(NXXF) BECAUSE ITER GREATER
     1THAN ",I8)
  113 CALL POTERS(IDET,NRHS,MAXN,AP,BP,CP,GP,PR,XP,CC,MT,T1,V1,MAX2,
     1IXPM,DETM,XNXX,LN,NJ,NW,NF,LP,DF)
      CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,1,LPRINT,WW,XNP,LP)
       IF(INXXFX.EQ.4)XPRES=XNP
       IF(INXXFX.NE.4)XNXX=XNP
       XCON(ITER)=ABS((XNP-XNN)/XNP)
      DO 115 J1=1,K1
      IF(WM(IMAX,J1).NE.0.) GO TO 57
      WCON(ITER,J1)=0.
      GO TO 115
   57 CONTINUE
      WCON(ITER,J1)=ABS((WM(IMAX,J1)-WWM(J1))/WM(IMAX,J1))
  115 CONTINUE
      WCH=WCON(ITER,JWMAX)
      IWH=JWMAX
      DO 116 J1=1,K2
      IF(FM(IMAX,J1).NE.0.) GO TO 58
      FCON(ITER,J1)=0.
      GO TO 116
   58 CONTINUE
      FCON(ITER,J1)=ABS((FM(IMAX,J1)-FFM(J1))/FM(IMAX,J1))
  116 CONTINUE
      FCH=FCON(ITER,JFMAX)
      IFH=JFMAX
      IF(LPRINT.NE.1) GO TO 117
       WRITE(6,118)ITER,WCH,FCH,WW,XNP,XCON(ITER)
       WRITE(6,119)(J1,WCON(ITER,J1),J1=1,K1)
       WRITE(6,119)(J1,FCON(ITER,J1),J1=1,K2)
  118  FORMAT(//,2X,"ITER=",I5,2X,"WCH=",E15.5,2X,"FCH=",E15.5,
     12X,"WW=",E15.5,2X,"XNP=",E15.5/,2X,"XCON(ITER)=",E15.5/)
  119    FORMAT(5(I5,E15.5))
  117  IF(XCON(ITER).GT.ECONV)GO TO 194
      IF(WCH.GT.ECONV) GO TO 194
```

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      IF(FCH.GT.CONV) GO TO 194
      GO TO 195
  194 IF(ITER.LE.4) GO TO 196
       IF(XCON(ITER).GT.XCON(ITER-1))GO TO 197
      IF(WCON(ITER,IWH).GT.WCON(ITER-1,IWH)) GO TO 197
      IF(FCON(ITER,IFH).GT.FCON(ITER-1,IFH)) GO TO 197
      GO TO 196
  197  WRITE(6,991)LP,JPS,WW,XNXX,XPRES,XNP
       WRITE(6,992)IWH,IFH,ITER,IASF,DWW,DDWW
  991  FORMAT(//,2X,"FOR W(",I3,2X,",",I3")=",E15.5,2X,"XNXX=",
     1E15.5,2X,"XPRES=",E15.5,2X,"XNP=",E15.5/)
  992  FORMAT(//,2X,"IWH=",I5,2X,"IFH=",I5,2X,"ITER=",I5//,4X,
     1"ITER",3X,"XCON(ITER)",3X,"WCON(ITER,IWH)",3X,"FCON(ITER,IFH
     2)",2X,"IASF=",I5,2X,"DWW=",E12.4,2X,"DDWW=",E12.4/)
       WRITE(6,993) (IK,XCON(IK),WCON(IK,IWH),FCON(IK,IFH),IK=1,
     1ITER)
  993  FORMAT(I8,3E15.5)
        IASF=IASF+1
        DDWW=DDWW/2.
         WW=SWW
        WW=WW+DDWW
        IF(IASF.GT.4)GO TO 403
        DO 216 I1=1,NLQPOT
        DO 217 J1=1,K1
        WM(I1,J1)=SWM(I1,J1)
        ETM(I1,J1)=SETM(I1,J1)
  217   WMP(I1,J1)=SWMP(I1,J1)
        DO 218 J1=1,K2
        FM(I1,J1)=SFM(I1,J1)
        XFM(I1,J1)=SXFM(I1,J1)
  218   FMP(I1,J1)=SFMP(I1,J1)
  216   CONTINUE
        XNP=SXNP
        GO TO 444
  196  XNN=XNP
      DO 131 J1=1,K1
  131 WWM(J1)=WM(IMAX,J1)
      DO 132 J1=1,K2
  132 FFM(J1)=FM(IMAX,J1)
      GO TO 333
  195   IVOUT=IVOUT+1
      CALL PCTSN(POT,POTM,STRY,STRA,1,1,1,XNXX)
      CALL SECOND(TIM2)
       TIM3=TIM2-TIM1
       TIM1=TIM2
       WRITE(6,241)LP,JPS,WW,XNXX,XPRES,NWAVE,ITER,TIM3
      WRITE(6,242)POT,POTM,STRY,STRA
      IF(IOUT.EQ.1)  WRITE(6,243)DETM,IXPM
      IF(LMOD.NE.1) GO TO 476
      CALL TRANSF(WF,XWF,FF,XFF,NW,NF,2,T1,MAXN,2,3,WW,XNP,LP)
  476   VOUT(1,IVOUT)=XNXX
        VOUT(2,IVOUT)=XPRES
        VOUT(3,IVOUT)=POT
        VOUT(4,IVOUT)=POTM
        VOUT(5,IVOUT)=STRY
        VOUT(6,IVOUT)=STRA
        VOUT(7,IVOUT)=WM(LP,1)
        VOUT(8,IVOUT)=WM(LP,2)
        NSHR=(NEQPOT+1)/2
        VOUT(9,IVOUT)=WM(NSHR,1)
        VOUT(10,IVOUT)=WM(NSHR,2)
        VOUT(11,IVOUT)=ITER
        VOUT(12,IVOUT)=NWAVE
  241    FORMAT(//,2X,"NONLINEAR SOLUTION FOR W(",I3,2X,",",I3")=",
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   1E15.5,2X,"NX=",E15.5,2X,"F=",E15.5/,2X,"MAXW                    AINE
   2D BY",I8,2X,"ITERATIONS"/,2X,"TIME COMPUTATION=",E12.4,2X,"SECONDS
   3"/,2X,"NNNXXXPPPNNNXXXPPPNNNXXXPPPNNNXXXPPPNNNXXXPPNNNXXXPPNNXXNN
   4PNNNXXXPPPNNNXXXPPPNNNXXXPPP"/,2X,"NNNXXXPPPNNNXXXPPPNNNXXXPNNXXPN
   5PPNNNXXXPPPNNNXXXPPPNNNXXXPPPNNNXXXPPPNNNXXXPPPNNNXXXPPP"/)
242  FORMAT(//,2X,"POTETIAL ENERGY=",E17.7,2X,"MODIFIED OF POTENTIAL
   1ENERGY=",E17.7/,2X,"END SHORTENING FOR Y=0.=",E17.7,2X,
   2"AVERAGE END SHORTENING=",E17.7/)
243   FORMAT(//,2X,"DETERMINANT=",E12.4,2X,"IXPM=",I10)
     IINN=IINN+1
      GO TO (231,232,233),IPATH
231   IF(XNP.GT.XNPS)GO TO 234
      IPATH=2
      GO TO 237
234   IF(XNP.LT.XNP1)GO TO 237
      WRITE(6,261)XNP,XNP1
261   FORMAT(//,2X,"END OF THIS CASE BECAUSE XNP=",E12.4,2X,
   1"ON THHE FIRST PATH IS GREATER THAN XNP1=",E12.4/)
      GO TO 403
232   IF(XNP.LE.XNPS)GO TO 237
      IPATH=3
      GO TC 237
233   IF(XNP.LT.XNPS)GO TO 237
      WRITE(6,262)XNP,XNP3
262   FORMAT(//,2X,"END OF THIS CASE BECAUSE XNP=",E12.4,2X,
   1"ON THE THIRD PATH IS GREATER THAN XNP3=",E12.4/)
      GO TO 403
237   XNPS=XNP
      IF(INON2.NE.2.OR.IINN.NE.IRR-5)GO TO 321
      WRITE(15,322)IINN,XNP
322   FORMAT(//,2X,"IINN=",I5,2X,"XNP=",E15.7)
      WRITE(15,323)((WM(I1,J1),I1=1,NEQPOT),J1=1,K1)
      WRITE(15,323)((ETM(I1,J1),I1=1,NEQPOT),J1=1,K1)
      WRITE(15,323)((WMP(I1,J1),I1=1,NEQPOT),J1=1,K1)
      WRITE(15,323)((FM(I1,J1),I1=1,NEQPOT),J1=1,K2)
      WRITE(15,323)((XFM(I1,J1),I1=1,NEQPOT),J1=1,K2)
      WRITE(15,323)((FMP(I1,J1),I1=1,NEQPOT),J1=1,K2)
323  FORMAT(5E16.8)
321   CONTINUE
     IF(IINN.LE.IRR)GO TO 721
     WRITE(6,722)IINN
     GO TO 403
722  FORMAT(//,2X,"END OF  THIS CASE BECAUSE  THE NUMBER OF DISPLACEME
   1ENT POITS IS GREATER THAN ",I8)
721     WW=WW+DWW
      IASF=0
      DOWW=DWW
     IF(INON.EQ.1)GO TO 555
      WM(LP,JPS)=WW
     GO TO 444
403 CONTINUE
     RETURN
     END
    SUBROUTINE IMPERF
    COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
    COMMON/CINTG/NEQPOT,MI(500)
    COMMON/FACTS/DL5,XL,XH
    COMMON/FIDFR/DELTA,AL1,GA1,AL2,BT2,GA2
    COMMON/PRES2/WZ(100,5),WZP(100,5),WZPP(100,5)
       COMMON/RVKA/XIMPS
    DV=1.
    PI=4.*ATAN(DV)
      AL1=XIMPS*XH
      AL2=1.1*XIMPS*XH
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      A1=PI/XL
      A2=2.*A1
    XX=0.
    DO 10 I=1,NEQPOT
      WZ(I,1)=-AC1*COS(A2*XX)
      WZ(I,2)=AC2*SIN(A1*XX)
      WZP(I,1)=AC1*A2*SIN(A2*XX)
      WZP(I,2)=AC2*A1*COS(A1*XX)
      WZPP(I,1)=AC1*A2*A2*COS(A2*XX)
      WZPP(I,2)=-AC2*A1*A1*SIN(A1*XX)
      IF(K1.LE.2)GO TO 50
        DO 11 J=3,K1
        WZ(I,J)=0.
        WZP(I,J)=0.
        WZPP(I,J)=0.
11      CONTINUE
50      CONTINUE
    XX=XX+DELTA
10  CONTINUE
    RETURN
    END
    SUBROUTINE TRANSF(WF,XWF,FF,XFF,NW,NF,NFF,T1,MAXN,IDER,IPRR,WW,XNS
   1,LP)
    COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
    COMMON/PRES3/FM(100,6),XFM(100,6),FMP(100,8)
    COMMON/FIDFR/DELTA,AL1,GA1,AL2,BT2,GA2
    COMMON/CDISK/I21(501),I22(501),I23(501)
    COMMON/CINTG/NEQPOT,MI(500)
    COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
     COMMON/NEWFT/JPS,INXXPX
    DIMENSION WF(NRF,NW),XWF(NRF,NW),FF(NRF,NF),XFF(NRF,NF),T1(MAXN)
    IF(IPRR.EQ.0) GO TO 278
    DO 10 I1=1,NEQPOT
    NL=MI(I1)
    CALL READMS(22,T1,NL,I1)
    IF(I1.NE.1) GO TO 175
    DO 11 J1=1,K1
    WF(1,J1)=T1(J1)
    WM(I1,J1)=T1(J1+K6)
    XWF(1,J1)=T1(J1+K3-1)
    ETM(I1,J1)=T1(J1+K3+K6-1)
11  CONTINUE
    DO 12 J1=1,K2
    FF(1,J1)=T1(J1+K1)
    FM(I1,J1)=T1(J1+K1+K6)
    XFF(1,J1)=T1(J1+K4)
    XFM(I1,J1)=T1(J1+K4+K6)
12  CONTINUE
    GO TO 10
175 DO 13 J1=1,K1
    WM(I1,J1)=T1(J1)
    ETM(I1,J1)=T1(J1+K3-1)
13  CONTINUE
    DO 14 J1=1,K2
    FM(I1,J1)=T1(J1+K1)
    XFM(I1,J1)=T1(J1+K4)
14  CONTINUE
    IF(I1.NE.NEQPOT) GO TO 10
     DO 15 J1=1,K1
    WF(2,J1)=T1(J1+K6)
    XWF(2,J1)=T1(J1+K3+K6-1)
15  CONTINUE
    DO 16 J1=1,K2
    FF(2,J1)=T1(J1+K1+K6)
```

```
      XFF(2,J1)=T1(J1+K4+K6)
   16 CONTINUE
   10 CONTINUE
       IF(LP.EQ.0.OR.INXXPX.EQ.1)GO TO 176
       XNS=WM(LP,JPS)
       WM(LP,JPS)=WW
  176  CONTINUE
      IF(IDER.NE.1) GO TO 275
      NEQP=NEQPOT-1
      DO 20 I1=1,NEQP
      DO 21 J1=1,K1
      WMP(I1,J1)=AL1*WM(I1-1,J1)+GA1*WM(I1+1,J1)
   21 CONTINUE
      DO 22 J1=1,K2
      FMP(I1,J1)=AL1*FM(I1-1,J1)+GA1*FM(I1+1,J1)
   22 CONTINUE
   20 CONTINUE
      DO 23 J1=1,K1
      WMP(1,J1)=AL1*WF(1,J1)+GA1*WM(2,J1)
      WMP(NEQPOT,J1)=AL1*WM(NEQP,J1)+GA1*WF(2,J1)
   23 CONTINUE
      DO 24 J1=1,K2
      FMP(1,J1)=AL1*FF(1,J1)+GA1*FM(2,J1)
      FMP(NEQPOT,J1)=AL1*FM(NEQP,J1)+GA1*FF(2,J1)
   24 CONTINUE
  275 IF(IPRR.NE.1)RETURN
  278 CONTINUE
      J1=0
      WRITE(6,400)J1
      WRITE(6,500)
      XX=0.
      WRITE(6,600)WF(1,1),XWF(1,1)
      DO 48 I1=1,NEQPOT
      WRITE(6,509)I1,XX,WM(I1,1),WMP(I1,1),ETM(I1,1)
      XX=XX+DELTA
   48 CONTINUE
      WRITE(6,600)WF(2,1),XWF(2,1)
      DO 49 J1=1,KFOUR
      WRITE(6,400)J1
      WRITE(6,500)
      WRITE(6,700)WF(1,J1+1),XWF(1,J1+1),FF(1,J1),XFF(1,J1)
      DO 51 I1=1,NEQPOT
      WRITE(6,609)I1,WM(I1,J1+1),WMP(I1,J1+1),ETM(I1,J1+1),FM(I1,J1),
     1FMP(I1,J1),XFM(I1,J1)
   51 CONTINUE
      WRITE(6,700)WF(2,J1+1),XWF(2,J1+1),FF(2,J1),XFF(2,J1)
   49 CONTINUE
      DO 52 J1=K1,K2
      WRITE(6,400)J1
      WRITE(6,500)
      WRITE(6,800)FF(1,J1),XFF(1,J1)
      DO 53 I1=1,NEQPOT
      WRITE(6,709)I1,FM(I1,J1),FMP(I1,J1),XFM(I1,J1)
   53 CONTINUE
      WRITE(6,800)FF(2,J1),XFF(2,J1)
   52 CONTINUE
  400 FORMAT(//,2X,"INTERMIDIAT RESULTS FOR KFOUR=",I8/2X,"*************
     1*******************************")
  500 FORMAT(//,2X,"POINT",4X,"LENGTH",10X,"W",14X,"WP",13X,"WPF",
     112X,"F",14X,"FP",13X,"FPP"/2X,"**************************************
     2*******************************************************
     3*******")
  509 FORMAT(I8,E17.4,3E15.6)
  600 FORMAT(//20H  FICTIVE  FOINT      E15.6,15X,E15.6//)
```

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  609 FORMAT(I6,12X,5E15.6)
  700 FORMAT(//20H   FICTIVE  POINT       E15.6,15X,2E15.6,15X,E15.6//)
  800 FORMAT(//65H   FICTIVE  POINT       *
     1                 E15.6,15X,F15.6//)
  709 FORMAT(I6,57X,3E15.6)
      RETURN
      END
      SUBROUTINE POTSN(POT,PCTM,STRY,STRA,IP,ISY,ISA,XNXX)
C    POT    - POTENTIAL ENERGY
C    STRY   -  UNIT END SHORTENING FOR Y=0.
C    STRA   -  AVERAGE UNIT END SHORTENING
C    IP=1   FOR CALCULATE POT
C ISY=1     FOR CALCULATE STRY
C    ISA=1 FOR CALCULATE STRA
      COMMON/FOURIN/KFOUR,K6,K4,K3,K2,K1
      COMMON/BOUND/LS1,LSN
      COMMON/GEOM/RR,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
      COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
      COMMON/PRES2/WZ(100,5),WZF(100,5),WZPP(100,5)
      COMMON/PRES3/FM(100,6),XFM(100,6),FMP(100,6)
      COMMON/FACTOR/C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12
      COMMON/FACT2/DL1,DL2,DL3,DL4,DA1,DA2,DA3,DA4,DB2,D33,D34,XNI,EXXP
      COMMON/FACT3/DL5,XL,XH
      COMMON/CINTG/NEQPCT,MI(500)
      COMMON/FIBER/DELTA,AL1,GA1,AL2,BT2,GA2
      COMMON/XXLOAD/XPRES
        COMMON/SHEINN/VCUT(12,130),VPOT(7,130),IVOUT
         DIMENSION PE(7),PEE(7)
      CE1=C10/2.
      CE2=C9**2/2.
      CE3=C9/((1.-XNI)*EXXP)
      CE4=C9*DD*(1.-XNI)
      POT=0.
         DO 553 INA=1,7
  553    PEE(INA)=0.
      STRY=0.
      STRA=0.
       DO 10 I1=1,NEQPOT
      E7=1.
      IF(I1.EQ.1.OR.I1.EQ.NEQPOT)E7=0.5
      E1=-Q11*ETM(I1,1)-1./RR*WM(I1,1)+DA2*XNXX
      E2=0.
      DO 11 J1=1,KFOUR
      JS=J1**2
      E2=JS*WM(I1,J1+1)*(WM(I1,J1+1)+2.*WZ(I1,J1+1))+E2
   11 CONTINUE
      E1=E1+CE1*E2
      IF(IP.NE.1) GO TO 100
        PE(1)=DB2/D11**2*E1**2
        PE(3)=DL1*ETM(I1,1)**2
        PE(4)=-XNXX*WMP(I1,1)*(WMP(I1,1)+2.*WZF(I1,1))+DA3*2.*ETM(
     1 I1,1)*XNXX
        PE(5)=-2.*XPRES*WM(I1,1)
        PE(7)=DB2/D11**2*DA2*DA2*XNXX*XNXX
        PE(6)=0.
  100 E1=E1*DA2/D11+DA3*ETM(I1,1)
      IF(ISY.NE.1) GO TO 110
      PSY=E1
  110 IF(ISA.NE.1) GO TO 120
      PSA=E1-WMP(I1,1)*(WMP(I1,1)+2.*WZP(I1,1))/2.
  120 IF(ISY.NE.1) GO TO 130
      E1=0.
      DO 16 J1=1,K1
      DO 16 J2=1,K1
```

```
      E1=[illegible]WMP(I1,J2)+2.*WZP(I1,[illegible]))+E1
   18 CONTINUE
      PSY=PSY-E1/2.
      E1=0.
      E2=0.
      DO 12 J1=1,KFOUR
       JS=J1**2
      E1=ETM(I1,J1+1)+E1
      E2=E2+JS*WM(I1,J1+1)
   12 CONTINUE
      PSY=PSY+DA3*E1-DA4*C9*E2
      E1=0.
      E2=0.
      DO 13 J1=1,K2
      JS=J1**2
      E1=E1+XFM(I1,J1)
      E2=E2+JS*FM(I1,J1)
   13 CONTINUE
      PSY=PSY+DA2*E1-DA1*C9*E2
      STRY=STRY+PSY*E7
  130 IF(ISA.NE.1) GO TO 140
      E1=0.
       DO 14 J1=1,KFOUR
   14 E1=E1+WMP(I1,J1+1)*(WMP(I1,J1+1)+2.*WZP(I1,J1+1))
      PSA=PSA-E1/4.
      STRA=STRA+PSA*E7
  140 IF(IP.NE.1) GO TO 10
      E1=0.
      E2=0.
       E3=0.
      E4=0.
       E5=0.
      DO 15 J1=1,KFOUR
      JS=J1**2
      JS2=JS**2
      E1=E1+WMP(I1,J1+1)*(WMP(I1,J1+1)+2.*WZP(I1,J1+1))
      E2=E2+JS2*WM(I1,J1+1)**2
      E3=E3+JS*WMP(I1,J1+1)**2
      E4=E4+ETM(I1,J1+1)**2
      E5=E5+JS*WM(I1,J1+1)*ETM(I1,J1+1)
   15 CONTINUE
        PE(3)=PE(3)+CE2*DL5*E2+CE4*E3+DL1*E4/2.-C9*DL2*E5
        PE(4)=PE(4)-XNXX*E1/2.
      E1=0.
      E2=0.
      E3=0.
      E4=0.
      DO 16 J1=1,K2
      JS=J1**2
      JS2=JS**2
      E1=JS2*FM(I1,J1)**2+E1
      E2=E2+JS*FMP(I1,J1)**2
      E3=E3+XFM(I1,J1)**2
      E4=E4+JS*FM(I1,J1)*XFM(I1,J1)
   16 CONTINUE
        PE(2)=CE2*DA1*E1+CE3*E2+DB2*E3/2.-DA2*C9*E4
        DO 143 INA=1,7
  143   PEE(INA)=PEE(INA)+PE(INA)*E7
   10 CONTINUE
       IF(IP.NE.1) GO TO 150
       DO 144 INA=1,7
  144  PEE(INA)=PEE(INA)*3.14159*RR*DELTA
       PEE(6)=-D22*3.14159*RR*XL*XNXX*XNXX
       POT=PEE(1)+PEE(2)+PEE(3)+PEE(4)+PEE(5)+PEE(6)
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      POTM=POT+PEE(6)+PEE(7)
       PEES=0.
       IF(LS1.GT.4.AND.LS1.NE.9)GO TO 253
       PEES=-DL3*6.283185*RR*XNXX*WMP(1,1)
  253  IF(LSN.GT.4.AND.LSN.NE.9)GO TO 254
       PEES=PEES+DL3*6.283185*RR*XNXX*WMP(NEQPCT,1)
  254  PEE(3)=PEE(3)+PEES
       POT=POT+PEES
       POTM=POTM+PEES
       DO 145 INA=1,7
  145  VPOT(INA,IVOUT)=PEE(INA)
  150 IF(ISY.NE.1) GO TO 160
      STRY=DA1*XNXX-STRY/XL*DELTA
  160 IF(ISA.NE.1) GO TO 170
      STRA=DA1*XNXX-STRA/XL*DELTA
  170 CONTINUE
      RETURN
       END
      SUBROUTINE ABCG(IEQ,M1,CF,BF,AF,GF,NRHS,XNXX,LN,NJ,NW,NF,LP,DF)
      COMMON/CINTG/NEQPOT,MI(500)
      COMMON/BOUND/LS1,LSN
      COMMON/FIDFR/DELTA,AL1,GA1,AL2,BT2,GA2
      COMMON/NEWPT/JPS,INXXPX
      COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
      COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
      DIMENSION AF(M1,M1),BF(M1,M1),CF(M1,M1),GF(M1,NRHS),DF(M1,1)
C   NEQPOT=NPOINT (EXCLUDING FICTIVES POINTS)
C LS1   -KIND OF BOUNDARY CONDITIONS OF POINT 1
C  LSN   -KIND OF BOUNDARY CONDITIONS OF POINT NP
C   NW  -  MAXIMUM  K+1  FOR DIMENSION  WM(NJ,NW)
C   NF  -  MAXIMUM 2*K  FOR DIMENSION  FM(NJ,NF)
      IF(IEQ.GT.1) GO TO 10
      CALL RSTG(BF,CF,AF,GF,1,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DF)
      DO 2 I1=1,K6
      GF(I1+K6,NRHS)=GF(I1,NRHS)
        DF(I1+K6,1)=DF(I1,1)
      DO 2 J1=1,K6
      BF(I1+K6,J1)=AL2*BF(I1,J1)+AL1*CF(I1,J1)
      BF(I1+K6,J1+K6)=BT2*BF(I1,J1)+AF(I1,J1)
      BF(I1,J1+K6)=GA2*BF(I1,J1)+GA1*CF(I1,J1)
    2 CONTINUE
      CALL BOUNDR(AF,CF,GF,1,XNXX,LS1,M1,NJ,NW,NF,LN,NRHS,DF)
      DO 3 I1=1,K6
      DO 3 J1=1,K6
      BF(I1,J1)=AL1*AF(I1,J1)
      AF(I1+K6,J1)=BF(I1,J1+K6)
      BF(I1,J1+K6)=CF(I1,J1)
      AF(I1,J1)=GA1*AF(I1,J1)
    3 CONTINUE
      RETURN
   10 IF(IEQ.GT.2) GO TO 20
      CALL RSTG(AF,CF,BF,GF,2,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DF)
      DO 4 I1=1,K6
      DO 4 J1=1,K6
      BF(I1,J1)=BF(I1,J1)+BT2*AF(I1,J1)
      CF(I1,J1+K6)=AL2*AF(I1,J1)+AL1*CF(I1,J1)
      AF(I1,J1)=GA2*AF(I1,J1)+GA1*CF(I1,J1)
      CF(I1,J1)=0.
    4 CONTINUE
      RETURN
   20 IF(IEQ.GE.NEQPOT-1) GO TO 30
      JP=IEQ
      CALL RSTG(AF,CF,BF,GF,JP,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DF)
      DO 5 I1=1,K6
```

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      DO 5 J1=1,K6
      BF(I1,J1)=BF(I1,J1)+BT2*AF(I1,J1)
      TEMP=GA2*AF(I1,J1)+GA1*CF(I1,J1)
      CF(I1,J1)=AL2*AF(I1,J1)+AL1*CF(I1,J1)
      AF(I1,J1)=TEMP
    5 CONTINUE
        IF(LP.EQ.0.OR.INXXPX.EQ.1)RETURN
        IF(IEQ.LT.LP-1.OR.IEQ.GT.LP+1)RETURN
        WWW=WM(LP,JPS)
        IF(IEQ.NE.LP-1)GO TO 51
        DO 53 I1=1,K6
        GF(I1,NRHS)=GF(I1,NRHS)-AF(I1,JPS)*WWW
   53   AF(I1,JPS)=DF(I1,1)
        RETURN
   51   IF(IEQ.NE.LP)GO TO 52
        DO 54 I1=1,K6
        GF(I1,NRHS)=GF(I1,NRHS)-BF(I1,JPS)*WWW
   54   BF(I1,JPS)=DF(I1,1)
        RETURN
   52   DO 55 I1=1,K6
        GF(I1,NRHS)=GF(I1,NRHS)-CF(I1,JPS)*WWW
   55   CF(I1,JPS)=DF(I1,1)
      RETURN
   30 IF(IEQ.EQ.NEQPOT) GO TO 40
      JP=IEQ
      CALL RSTG(AF,CF,BF,GF,JP,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DF)
      DO 6 I1=1,K6
      DO 6 J1=1,K6
      BF(I1,J1)=BF(I1,J1)+BT2*AF(I1,J1)
      TEMP=GA2*AF(I1,J1)+GA1*CF(I1,J1)
      CF(I1,J1)=AL2*AF(I1,J1)+AL1*CF(I1,J1)
      AF(I1,J1)=TEMP
      AF(I1,J1+K6)=0.
    6 CONTINUE
      RETURN
   40 JP=IEQ
      CALL RSTG(AF,CF,BF,GF,JP,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DF)
      DO 7 I1=1,K6
      GF(I1+K6,NRHS)=GF(I1,NRHS)
       DF(I1+K6,1)=DF(I1,1)
      DO 7 J1=1,K6
      BF(I1,J1)=BF(I1,J1)+BT2*AF(I1,J1)
      BF(I1,J1+K6)=GA2*AF(I1,J1)+GA1*CF(I1,J1)
      BF(I1+K6,J1)=AL2*AF(I1,J1)+AL1*CF(I1,J1)
    7 CONTINUE
      CALL BOUNDR(CF,AF,GF,JP,XNXX,LSN,M1,NJ,NW,NF,LN,NRHS,DF)
      DO 8 I1=1,K6
      TEMP=GF(I1,NRHS)
       TEMPP=DF(I1,1)
      GF(I1,NRHS)=GF(I1+K6,NRHS)
      GF(I1+K6,NRHS)=TEMP
        DF(I1,1)=DF(I1+K6,1)
        DF(I1+K6,1)=TEMPP
      DO 8 J1=1,K6
      BF(I1+K6,J1+K6)=GA1*CF(I1,J1)
      CF(I1+K6,J1)=AL1*CF(I1,J1)
      CF(I1,J1)=BF(I1+K6,J1)
      BF(I1+K6,J1)=AF(I1,J1)
    8 CONTINUE
      RETURN
      END
      FUNCTION ALB(I,J,L,B,JP,N2,N3,N4,LL)
C     N4=1 FOR   B(JP,I+J) OR B(JP,I-J)
C     N4=2  FOR B(JP,J)
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C   LL=1 FOR W    LL=2  FOR F
      COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
      DIMENSION B(N2,N3)
      IF(L.GT.3) GO TO 10
      I1=I+J
      I2=I1
      GO TO 20
   10 I1=IABS(I-J)
      I2=I1
   20 IF(N4.EQ.1) GO TO 120
      I2=J
      GO TO 100
  120 IF(I1.LE.KFOUR) GO TO 100
      ALB=0.
      RETURN
  100 IF(L.LE.3) GO TO 110
      ETA=1.
      IF(I.EQ.J) ETA=0.
  110 GO TO(11,12,13,14,15,16),L
   11 R1=I1**2
      GO TO 17
   12 R1=J**2
      GO TO 17
   13 R1=2.*I1*J
      GO TO 17
   14 R1=(2.-ETA)*I1**2
      GO TO 17
   15 R1=(2.-ETA)*J**2
      GO TO 17
   16 IF(I-J.LT.0) ETA=-1.
      R1=-2.*ETA*J*I1
   17 IF(LL.EQ.1)I2=I2+1
      ALB=R1*B(JP,I2)
      RETURN
      END
      SUBROUTINE RSTG(R,S,T,G,JF,XNXX,M1,NJ,NW,NF,LN,NRHS,LP,DE)
C   LN=1   FOR LINEAR     LN=2  FOR NONLINEAR
C   NJ        MAXIMUM  POINTS IN MERIDIONAL DIRECTION
C   NW     MAXIMUM  KFOUR+1        NF-  MAXIMUM 2.*KFOUR
C   XNXX    AXIAL COMPRESSION LOAD   JP-  THE POINT  IN MERIDIONAL DIRECT
      COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
      COMMON/GEOM/RR,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
      COMMON/FACTOR/C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12
      COMMON/FACT2/DL1,DL2,DL3,DL4,DA1,DA2,DA3,DA4,DB2,DB3,DB4,XNI,EXXP
      COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
      COMMON/PRES2/WZ(100,5),WZP(100,5),WZPP(100,5)
      COMMON/PRES3/FM(100,8),XFM(100,8),FMP(100,8)
      COMMON/XXLOAD/XFRES
       COMMON/NEWPT/JFS,INXXFX
      DIMENSION R(M1,M1),S(M1,M1),T(M1,M1),G(M1,NRHS)
        DIMENSION DE(M1,1)
C   INXXPX=1 FOR NXX AND P AS KNOWNS
C   INXXPX=2 FOR NXX AS UNKNOWN THE INITIAL SOLUTION IS NXX=0.
C   INXXPX=3 SAME AS 2 BUT THE INITIAL SOLUTION IS A GIVEN NXX
C   INXXPX=4 FOR P AS UNKNOWN
C     K6=6*KFOUR+2
C   K4=4*KFOUR+2
C   K3=3*KFOUR+2
C   K2=2*KFOUR
C   K1=KFOUR+1
C   C9=(NNN/RR)**2
C     C1=DD*H11
C    C2=2.*DD*H12*C9
C    C3=2.*D12*C9
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C     C4=DD*H22*C9**2
C      C5=Q11/D11*C9
C     C6=1./(RR*D11)*C9
C     C7=C9**2/(2.*D11)
C      C8=Q22*C9**2
C     C10=C9/2.
C     C11=2.*C9*D12
C     C12=D22*C9**2
          RCOR1=DA2/(RR*D11)
          RCOR2=C9*DA2/D11
      DO 1 I1=1,K6
      G(I1,NRHS)=0.
        DE(I1,1)=0.
      DO 1 J1=1,K6
      R(I1,J1)=0.
      S(I1,J1)=0.
      T(I1,J1)=0.
    1 CONTINUE
      J1=0
      DO 2 I1=K3,K6
       T(I1,I1)=1.
      J1=J1+1
      R(I1,J1)=-1.
    2 CONTINUE
C   EQUILIBRIUM EQUATION  FOR I=0
      R(1,K3)=DD*H11+Q11**2/D11
      T(1,K3)=2.*Q11/(RR*D11)
      T(1,1)=1./(RR**2*D11)
        GO TO(71,72,73,74),INXXPX
   71   T(1,K3)=T(1,K3)+XNXX
        G(1,NRHS)=-XNXX*WZPP(JP,1)+XPRES+RCOR1*XNXX
        GO TO 75
   72   DE(1,1)=WZPP(JP,1)-RCOR1
        G(1,NRHS)=XPRES
        IF(LN.EQ.1) GO TO 75
        DE(1,1)=DE(1,1)+ETM(JP,1)
        G(1,NRHS)=G(1,NRHS)+XNXX*ETM(JP,1)
        T(1,K3)=T(1,K3)+XNXX
        GO TO 75
   73   T(1,K3)=T(1,K3)+XNXX
        DE(1,1)=WZPP(JP,1)-RCOR1
        G(1,NRHS)=XPRES
        IF(LN.EQ.1) GO TO 75
        DE(1,1)=DE(1,1)+ETM(JP,1)
        G(1,NRHS)=G(1,NRHS)+XNXX*ETM(JP,1)
        GO TO 75
   74   T(1,K3)=T(1,K3)+XNXX
        G(1,NRHS)=-XNXX*WZPP(JP,1)+RCOR1*XNXX
        DE(1,1)=-1.
   75 DO 6 J=1,KFOUR
      IS=J**2
      M=J+1
      T(1,K3+J)=T(1,K3+J)-C5/2.*IS*WZ(JP,M)
      S(1,J+1)=S(1,J+1)-C5*IS*WZP(JP,M)
      T(1,J+1)=T(1,J+1)-IS/2.*(C5*WZPP(JP,M)+C6*WZ(JP,M))
      T(1,K4+J)=T(1,K4+J)+C10*IS*WZ(JP,M)
      T(1,K1+J)=T(1,K1+J)+C10*IS*WZPP(JP,M)
      S(1,K1+J)=S(1,K1+J)+2.*C10*IS*WZP(JP,M)
        IF(LN.EQ.2.OR.INXXPX.EQ.1)GO TO 94
        IF(JP.NE.LP.OR.JPS.NE.M)GO TO 94
        T(1,K3+J)=T(1,K3+J)-C5/2.*IS*WM(JP,M)
        T(1,J+1)=T(1,J+1)-IS/2.*C6*WM(JP,M)
        T(1,K4+J)=T(1,K4+J)+C10*IS*WM(JP,M)
        G(1,NRHS)=G(1,NRHS)-C6/4.*IS*WM(JP,M)**2
```

```
   94 IF(LN.EQ.1) GO TO 6
      T(1,K3+J)=T(1,K3+J)-C5/2.*IS*WM(JP,M)+C10*IS*FM(JP,J)
      T(1,J+1)=T(1,J+1)-IS/2.*(C5*ETM(JP,M)+C6*WM(JP,M))
     1 +C10*IS*XFM(JP,J)
      S(1,J+1)=S(1,J+1)-C5*IS*WMP(JP,M)+2.*C10*IS*FMP(JP,J)
      T(1,K4+J)=T(1,K4+J)+C10*IS*WM(JP,M)
      T(1,K1+J)=T(1,K1+J)+C10*IS*ETM(JP,M)
      S(1,K1+J)=S(1,K1+J)+2.*C10*IS*WMP(JP,M)
      G(1,NRHS)=G(1,NRHS)-C5/2.*IS*(WM(JP,M)*ETM(JP,M)+WMP(JP,M)**2)
     1-C6/4.*IS*(WM(JP,M)**2)+C10*IS*(WM(JP,M)*XFM(JP,J)+ETM(JP,M)*
     2FM(JP,J)+2.*WMP(JP,M)*FMP(JP,J))
    6 CONTINUE
C  EQUILIBRIUM EQUATIONS FOR I=1,2,,,,,,,KFOUR
      DO 3 I1=2,K1
      I=I1-1
      IS=I**2
      IS2=IS**2
      T(I1,1)=-C6*IS*WZ(JP,I1)
      T(I1,K3)=-C5*IS*WZ(JP,I1)
      T(I1,I1)=C4*IS2
      T(I1,I1+K1-1)=-C8*IS2
      T(I1,I1+K3-1)=-C2*IS
      T(I1,I1+K4-1)=C3*IS-1./RR
      R(I1,I1+K3-1)=C1
      R(I1,I1+K4-1)=-Q11
      E1=C7*IS*WZ(JP,I1)
        GO TO (76,77,78,76),INXXPX
   76   T(I1,I1+K3-1)=T(I1,I1+K3-1)+XNXX
        G(I1,NRHS)=-XNXX*WZPP(JP,I1)-RCOR2*IS*XNXX*WZ(JP,I1)
         T(I1,I1)=T(I1,I1)+RCOR2*IS*XNXX
        GO TO 831
   77   DE(I1,1)=WZPP(JP,I1)+RCOR2*IS*WZ(JP,I1)
        IF(LN.EQ.1) GO TO 831
        DE(I1,1)=DE(I1,1)+ETM(JP,I1)+RCOR2*IS*WM(JP,I1)
        G(I1,NRHS)=G(I1,NRHS)+XNXX*ETM(JP,I1)+RCOR2*IS*WM(JP,I1)*XNXX
        T(I1,I1+K3-1)=T(I1,I1+K3-1)+XNXX
        T(I1,I1)=T(I1,I1)+RCOR2*IS*XNXX
        GO TO 831
   78   T(I1,I1+K3-1)=T(I1,I1+K3-1)+XNXX
        DE(I1,1)=WZPP(JP,I1)+RCOR2*IS*WZ(JP,I1)
        T(I1,I1)=T(I1,I1)+RCOR2*IS*XNXX
        IF(LN.EQ.1) GO TO 831
        G(I1,NRHS)=G(I1,NRHS)+XNXX*ETM(JP,I1)+RCOR2*IS*XNXX*WM(JP,I1)
        DE(I1,1)=DE(I1,1)+ETM(JP,I1)+RCOR2*IS*WM(JP,I1)
  831   CONTINUE
        IF(LN.EQ.2.OR.INXXPX.EQ.1)GO TO 104
        IF(JP.NE.LP.OR.JPS.NE.I1)GO TO 104
        T(I1,1)=T(I1,1)-C6*IS*WM(JP,I1)
        T(I1,K3)=T(I1,K3)-C5*IS*WM(JP,I1)
        E1=C7*IS*(WM(JP,I1)+WZ(JP,I1))
        E2=C7/2.*IS
        E3=E2*WM(JP,I1)
  104 IF(LN.EQ.1) GO TO 60
      T(I1,1)=T(I1,1)-C6*IS*WM(JP,I1)
      T(I1,K3)=T(I1,K3)-C5*IS*WM(JP,I1)
      T(I1,I1)=T(I1,I1)-IS*(C5*ETM(JP,1)+C6*WM(JP,1))
      E1=C7*IS*(WM(JP,I1)+WZ(JP,I1))
      E2=C7/2.*IS
      E3=E2*WM(JP,I1)
      G(I1,NRHS)=G(I1,NRHS)-C5*IS*ETM(JP,1)*WM(JP,I1)
     1 -C6*IS*WM(JP,1)*WM(JP,I1)
   60 DO 4 J=1,KFOUR
      JS=J**2
       M=J+1
```

```
      T(I1,J+1)=T(I1,J+1)+E1*JS*WZ(JP,M)
        IF(LN.EQ.1.AND.INXXPX.EQ.1)GO TO 4
        IF(LN.EQ.1.AND.JP.NE.LP)GO TO 4
        IF(LN.EQ.1.AND.JPS.NE.M)GO TO 4
      T(I1,J+1)=T(I1,J+1)+E1*JS*WM(JP,M)
      T(I1,I1)=T(I1,I1)+E2*JS*WM(JP,M)*(WM(JP,M)+2.*WZ(JP,M))
      G(I1,NRHS)=G(I1,NRHS)+E1/2.*JS*WM(JP,M)**2+E3*JS*WM(JP,M)
     1*(WM(JP,M)+2.*WZ(JP,M))
    4 CONTINUE
      DO 5 J=1,K2
      T(I1,K4+J)=T(I1,K4+J)+C10*(ALB(I,J,1,WZ,JP,NJ,NW,1,1)+
     1ALB(I,J,4,WZ,JP,NJ,NW,1,1))
      T(I1,K1+J)=T(I1,K1+J)+C10*(ALB(I,J,2,WZPP,JP,NJ,NW,1,1)+
     1 ALB(I,J,5,WZPP,JP,NJ,NW,1,1))
      S(I1,K1+J)=S(I1,K1+J)+C10*(ALB(I,J,3,WZP,JP,NJ,NW,1,1)+
     1 ALB(I,J,6,WZP,JP,NJ,NW,1,1))
        IF(LN.EQ.2.OR.INXXPX.EQ.1)GO TO 107
        IF(JP.NE.LP)GO TO 107
        IF(JPS.NE.I+J+1)GO TO 108
        T(I1,K4+J)=T(I1,K4+J)+C10*ALB(I,J,1,WM,JP,NJ,NW,1,1)
  108   IF(JPS.NE.IABS(I-J)+1)GO TO 107
        T(I1,K4+J)=T(I1,K4+J)+C10*ALB(I,J,4,WM,JP,NJ,NW,1,1)
  107 IF(LN.EQ.1) GO TO 5
      T(I1,K4+J)=T(I1,K4+J)+C10*(ALB(I,J,1,WM,JP,NJ,NW,1,1)+
     1 ALB(I,J,4,WM,JP,NJ,NW,1,1))
      T(I1,K1+J)=T(I1,K1+J)+C10*(ALB(I,J,2,ETM,JP,NJ,NW,1,1)+
     1 ALB(I,J,5,ETM,JP,NJ,NW,1,1))
      S(I1,K1+J)=S(I1,K1+J)+C10*(ALB(I,J,3,WMP,JP,NJ,NW,1,1)+
     1 ALB(I,J,6,WMP,JP,NJ,NW,1,1))
      G(I1,NRHS)=G(I1,NRHS)+C10*((ALB(I,J,1,WM,JP,NJ,NW,1,1)+
     1 ALB(I,J,4,WM,JP,NJ,NW,1,1))*XFM(JP,J)+(ALB(I,J,2,ETM,JP,NJ,NW,
     21,1)+ALB(I,J,5,ETM,JP,NJ,NW,1,1))*FM(JP,J)+(ALB(I,J,3,WMP,JP,NJ,
     3 NW,1,1)+ALB(I,J,6,WMP,JP,NJ,NW,1,1))*FMP(JP,J))
      IJ1=I+J
      IF(IJ1.GT.KFOUR) GO TO 80
      T(I1,IJ1+1)=T(I1,IJ1+1)+C10*(ALB(I,J,1,XFM,JP,NJ,NF,2,2))
      T(I1,K3+IJ1)=T(I1,K3+IJ1)+C10*ALB(I,J,2,FM,JP,NJ,NF,2,2)
      S(I1,IJ1+1)=S(I1,IJ1+1)+C10*ALB(I,J,3,FMP,JP,NJ,NF,2,2)
   80 IJ2=IABS(I-J)
      IF(IJ2.GT.KFOUR) GO TO 5
      T(I1,IJ2+1)=T(I1,IJ2+1)+C10*ALB(I,J,4,XFM,JP,NJ,NF,2,2)
      T(I1,K3+IJ2)=T(I1,K3+IJ2)+C10*ALB(I,J,5,FM,JP,NJ,NF,2,2)
      S(I1,IJ2+1)=S(I1,IJ2+1)+C10*ALB(I,J,6,FMP,JP,NJ,NF,2,2)
    5 CONTINUE
    3 CONTINUE
C     COMPATIBILITY EQUATIONS FOR I=1,2,....,2KFOUR
      E1=C10/2.
      I2=K1+1
      I3=I2+K2-1
      DO 7 I1=I2,I3
      I=I1-K1
      IS=I**2
      R(I1,I1+K4-K1)=D11
      T(I1,I1+K4-K1)=-IS*C11
      T(I1,I1)=IS**2*C12
      IF(I.GT.KFOUR) GO TO 82
      R(I1,I1+K3-K1)=D11
      T(I1,I1+K3-K1)=-IS*C5+1./AR
      T(I1,I1-K1+1)=IS**2*C8
   82 DO 9 J1=2,K1
      J=J1-1
      T(I1,J1+K3-1)=T(I1,J1+K3-1)-C10*(ALB(I,J,1,WZ,JP,NJ,NW,1,1)+
     1ALB(I,J,4,WZ,JP,NJ,NW,1,1))
      T(I1,J1)=T(I1,J1)-C10*(ALB(I,J,2,WZPP,JP,NJ,NW,1,1)+
```

```
      1 ALB(I,J,5,WZPP,JP,NJ,NW,1,1))
       S(I1,J1)=S(I1,J1)-C10*(ALB(I,J,3,WZP,JP,NJ,NW,1,1)+
      1 ALB(I,J,6,WZP,JP,NJ,NW,1,1))
        IF(LN.EQ.2.OR.INXXPX.EQ.1)GO TO 109
        IF(JP.NE.LP)GO TO 109
        IF(JPS.NE.I+J+1)GO TO 111
        T(I1,J1+K3-1)=T(I1,J1+K3-1)-E1*ALB(I,J,1,WM,JP,NJ,NW,1,1)
  111   IF(JPS.NE.IABS(I-J)+1)GO TO 112
        T(I1,J1+K3-1)=T(I1,J1+K3-1)-E1*ALB(I,J,4,WM,JP,NJ,NW,1,1)
  112   IJ3=I+J
        IF(IJ3.GT.KFOUR)GO TO 113
        IF(JPS.NE.J+1)GO TO 113
        T(I1,K3+IJ3)=T(I1,K3+IJ3)-E1*ALB(I,J,2,WM,JP,NJ,NW,2,1)
  113   IJ3=IABS(I-J)
        IF(IJ3.GT.KFOUR)GO TO 109
        IF(JPS.NE.J+1)GO TO 109
        T(I1,K3+IJ3)=T(I1,K3+IJ3)-E1*ALB(I,J,5,WM,JP,NJ,NW,2,1)
  109 IF(LN.EQ.1) GO TO 9
       T(I1,J1+K3-1)=T(I1,J1+K3-1)-E1*(ALB(I,J,1,WM,JP,NJ,NW,1,1)+
      1 ALB(I,J,4,WM,JP,NJ,NW,1,1))
       T(I1,J1)=T(I1,J1)-E1*(ALB(I,J,2,ETM,JP,NJ,NW,1,1)+
      1 ALB(I,J,5,ETM,JP,NJ,NW,1,1))
       S(I1,J1)=S(I1,J1)-E1*(ALB(I,J,3,WMP,JP,NJ,NW,1,1)+
      1 ALB(I,J,6,WMP,JP,NJ,NW,1,1))
       G(I1,NRHS)=G(I1,NRHS)-E1*((ALB(I,J,1,WM,JP,NJ,NW,1,1)
      1+ALB(I,J,4,WM,JP,NJ,
      2NW,1,1))*ETM(JP,J1)+(ALB(I,J,2,ETM,JP,NJ,NW,1,1) +ALB(I,J,5,ETM,
      3JP,NJ,NW,1,1))*WM(JP,J1)+(ALB(I,J,3,WMP,JP,NJ,NW,1,1)+ALB(I,J,6,
      4WMP,JP,NJ,NW,1,1))*WMF(JP,J1))
       IJ1=I+J
       IF(IJ1.GT.KFOUR) GO TO 83
       T(I1,IJ1+1)=T(I1,IJ1+1)-E1*(ALB(I,J,1,ETM,JP,NJ,NW,2,1))
       T(I1,K3+IJ1)=T(I1,K3+IJ1)-E1*(ALB(I,J,2,WM,JP,NJ,NW,2,1))
       S(I1,IJ1+1)=S(I1,IJ1+1)-E1*ALB(I,J,3,WMP,JP,NJ,NW,2,1)
   83 IJ2=IABS(I-J)
       IF(IJ2.GT.KFOUR) GO TO 9
       T(I1,IJ2+1)=T(I1,IJ2+1)-E1*ALB(I,J,4,ETM,JP,NJ,NW,2,1)
       T(I1,K3+IJ2)=T(I1,K3+IJ2)-E1*ALB(I,J,5,WM,JP,NJ,NW,2,1)
       S(I1,IJ2+1)=S(I1,IJ2+1)-E1*ALB(I,J,6,WMP,JP,NJ,NW,2,1)
    9 CONTINUE
    7 CONTINUE
       RETURN
       END
       SUBROUTINE BOUNDR(BS,BT,BG,IN,XNXX,LS,M1,NJ,NW,NF,LN,NRHS,DE)
       COMMON/FOURIR/KFOUR,K5,K4,K3,K2,K1
       COMMON/PRES1/WM(100,5),ETM(100,5),WMP(100,5)
       COMMON/PRES2/WZ(100,5),WZP(100,5),WZPP(100,5)
       COMMON/PRES3/FM(100,8),XFM(100,8),FMP(100,8)
       COMMON/GEOM/AB,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
       COMMON/FACT2/DL1,DL2,DL3,DL4,DA1,DA2,DA3,DA4,DB2,DB3,DB4,XNI,EXXP
       COMMON/FACTOR/C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12
        COMMON/NEWPT/JPS,INXXPX
       DIMENSION BS(M1,M1),BT(M1,M1),BG(M1,NRHS),DE(M1,1)
C     BOUNDARY CONDITIONS  BS*Z'+BT*Z=BG
C   LS=1 FOR  SS1    W=MX=NXY=NX=0.
C  LS=2 FOR SS2   W=MX=NXY=U=0.
C   LS=3 FOR SS3    W=MX=V=NX=0.
C   LS=4  FOR SS4   W=MX=V=U=0.
C   LS=5 FOR CC1    W=W,X=NXY=NX=0.
C   LS=6  FOR CC2   W=W,X=NXY=U=0.
C    LS=7  FOR CC3    W=W,X=V=NX=0.
C    LS=8  FOR CC4    W=W,X=V=U=0.
C   LS=9 FOR FREE EDGE   NX=NXY=QX=MX=0.
C   LS=10 FOR SYMMETRY    NXY=QX=W,X=U=0.
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C     LS=11 FOR ANTISYMMETRY   NX=MX=W=V=0.
C     DALFA=(1.+XLAMD)*(1.+YLAMD)-XNI**2
C     DL1=D0*(1.+RHOX+EX**2*XLAMD*(1.+YLAMD-XNI**2))*12./(XH**2*DALFA))
C     DL2=D0*XNI*(1.+EX*EY*YLAMD*XLAMD*12./(XH**2*DALFA))
C     DL3=EX*XLAMD*(1.+YLAMD)/DALFA
C     DL4=-XNI*EX*XLAMD/DALFA
C    DA1=(1.+YLAMD)/(DALFA*EXXP)
C     DA2=-XNI/(DALFA*EXXP)
C     DA3=-(1.+YLAMD)*EX*XLAMD/DALFA
C     DA4=XNI*EY*YLAMD/DALFA
C    DB2=(1.+XLAMD)/(DALFA*EXXP)
C     DB3=XNI*EX*XLAMD/DALFA
C     DB4=-(1.+XLAMD)*EY*YLAMD/DALFA
C       K6=6*KFOUR+2
C      K4=4*KFOUR+2
C      K3=3*KFOUR+2
C      K2=2*KFOUR
C      K1=KFOUR+1
         RCK=1.
           C80T=(0.-DL4*DA2/D11)/(DL1-DL4*Q11/D11)
C       CORRECTED IN 1980 GA TECH
        IF(LS.EQ.11)LS=3
        DO 1 I1=1,K6
        BG(I1,NRHS)=0.
         DE(I1,1)=0.
        DO 1 J1=1,K6
        BS(I1,J1)=0.
        BT(I1,J1)=0.
    1 CONTINUE
        IF(LS.EQ.10) GO TO 888
        IF(LS.GT.8) GO TO 100
        DO 2 I1=1,K1
    2 BT(I1,I1)=1.
        IF(LS.LE.4.OR.LS.GT.8) GO TO 100
  888 J=0
        DO 3 I1=K3,K4
        J=J+1
    3 BS(I1,J)=1.
  100 IF(LS.GT.4) GO TO 200
        BT(K3,K3)=1.
         IF(INXXPX.EQ.1.OR.INXXPX.EQ.4)GO TO 77
         DE(K3,1)=-C80T*RCK
         BG(K3,1)=0.
         GO TO 78
   77    BG(K3,NRHS)=C80T*XNXX*RCK
   78    CONTINUE
        K31=K3+1
        DO 4 I1=K31,K4
        BT(I1,I1)=DL1
        BT(I1,I1+K4-K3)=DL4
        IF(LS.EQ.1.OR.LS.EQ.3) GO TO 4
        I=I1-K3
        BT(I1,I1+K1-K3)=BT(I1,I1+K1-K3)-I**2*C9*DL3
    4 CONTINUE
  200 IF(LS.NE.10) GO TO 300
        E1=DL4/D11*C10
        BS(1,K3)=DL1-DL4/D11*Q11
        DO 5 J1=1,KFOUR
        JS=J1**2
        BS(1,J1+1)=BS(1,J1+1)+E1*JS*WZ(IN,J1+1)
        BT(1,J1+1)=BT(1,J1+1)+E1*JS*WZP(IN,J1+1)
        BT(1,K1+J1)=BT(1,K1+J1)+C10*JS*WZP(IN,J1+1)
        IF(LN.EQ.1) GO TO 5
        BS(1,J1+1)=BS(1,J1+1)+E1*JS*WM(IN,J1+1)
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      BT(1,J1+1)=BT(1,J1+1)+E1*JS*WMP(IN,J1+1)
      BG(1,NRHS)=BG(1,NRHS)+E1*JS*WM(IN,J1+1)*WMP(IN,J1+1)
    5 CONTINUE
      DO 12 I1=2,K1
      I=I1-1
      BS(I1,I1+K3-1)=BS(I1,I1+K3-1)+DL1
      BS(I1,I1+K4-1)=BS(I1,I1+K4-1)+DL4
      DO 12 J1=1,K2
      BT(I1,K1+J1)=BT(I1,K1+J1)+Q10*(ALB(I,J1,2,WZP,IN,NJ,NW,1,1)+
     1 ALB(I,J1,5,WZP,IN,NJ,NW,1,1))
   12 CONTINUE
  300 IF(LS.NE.9) GO TO 400
      E1=DL4/D11*Q11
      BT(1,K3)=DL1-E1
      BS(K3,K3)=DL1-E1
      E1=DL4/(D11*RR)
      BT(1,1)=-E1
        IF(INXXPX.EQ.1.OR.INXXPX.EQ.4)GO TO 87
        DE(1,1)=-(0.-DL4*DA2/D11)*RCK
        DE(K3,1)=WZP(IN,1)
        BS(K3,1)=-E1
        IF(INXXPX.EQ.3) BS(K3,1)=XNXX-E1
        IF(LN.EQ.1) GO TO 88
        BS(K3,1)=XNXX-E1
        DE(K3,1)=+WMP(IN,1)+DE(K3,1)
        BG(K3,1)=BG(K3,1)-WMP(IN,1)*XNXX
        GO TO 88
   87   BS(K3,1)=XNXX-E1
      BG(K3,NRHS)=-XNXX*WZP(IN,1)
           BG(1,NRHS)=(0.-DL4*DA2/D11)*XNXX*RCK
   88  CONTINUE
      E1=DL4/D11*C10
      DO 6 J1=1,KFOUR
      JS=J1**2
      BT(1,J1+1)=BT(1,J1+1)+E1*JS*WZ(IN,J1+1)
      BT(K3,J1+1)=BT(K3,J1+1)+E1*JS*WZP(IN,J1+1)
      BS(K3,J1+1)=BS(K3,J1+1)+E1*JS*WZ(IN,J1+1)
      IF(LN.EQ.1) GO TO 6
      BT(1,J1+1)=BT(1,J1+1)+E1*JS*WM(IN,J1+1)
      BG(1,NRHS)=BG(1,NRHS)+E1/2.*JS*WM(IN,J1+1)**2
      BT(K3,J1+1)=BT(K3,J1+1)+E1*JS*WMP(IN,J1+1)
      BS(K3,J1+1)=BS(K3,J1+1)+E1*JS*WM(IN,J1+1)
      BG(K3,NRHS)=BG(K3,NRHS)+E1*JS*WM(IN,J1+1)*WMP(IN,J1+1)
    6 CONTINUE
      DO 13 I1=2,K1
      I=I1-1
      IS=I**2
      BT(I1,I1+K3-1)=DL1
      BT(I1,I1)=-IS*C9*DL2
       BT(I1,I1+K4-1)=DL4
      BS(I1+K3-1,I1+K4-1)=DL4
      BS(I1+K3-1,I1+K3-1)=DL1
        IF(INXXPX.EQ.2.OR.INXXPX.EQ.3)GO TO 97
      BS(I1+K3-1,I1)=-IS*C9*(DL2+2.*DD*(1.-XNI))+XNXX
      BG(I1+K3-1,NRHS)=-XNXX*WZP(IN,I1)
        GO TO 13
   97   BS(I1+K3-1,I1)=-IS*C9*(DL2+2.*DD*(1.-XNI))
        IF(INXXPX.EQ.3)BS(I1+K3-1,I1)=BS(I1+K3-1,I1)+XNXX
        DE(I1+K3-1,1)=WZP(IN,I1)
        IF(LN.EQ.1)GO TO 13
        DE(I1+K3-1,1)=DE(I1+K3-1,1)+WMP(IN,I1)
        IF(INXXPX.EQ.2)BS(I1+K3-1,I1)=BS(I1+K3-1,I1)+XNXX
        BG(I1+K3-1,1)=BG(I1+K3-1,1)-XNXX*WMP(IN,I1)
   13 CONTINUE
```

```
400 I2=K1+1
    I3=I2+K2-1
    I4=K3-1
    DO 7 I1=I2,I3
    I=I1-K1
    IS=I**2
    GO TO (21,22,23,24,21,26,27,28,21,30),LS
 21 BS(I1,K1+I)=1.
    BT(I1+I4,K1+I)=1.
    GO TO 7
 22 BS(I1,K1+I)=1.
    BS(I1+I4,K4+I)=DB2
    DO 8 J1=1,K1
    J=J1-1
    BS(I1+I4,J1)=BS(I1+I4,J1)-C10*(ALB(I,J,1,WZ,IN,NJ,NW,1,1)+
   1ALB(I,J,4,WZ,IN,NJ,NW,1,1))
  8 CONTINUE
    IF(I.GT.KFOUR) GO TO 7
    BS(I1+I4,K3+I)=BS(I1+I4,K3+I)+DB3
    BS(I1+I4,I+1)=BS(I1+I4,I+1)-IS*C9*DB4+1./RR
    GO TO 7
 23 BT(I1,K1+I)=1.
    BT(I1+I4,K4+I)=DB2
    IF(I.GT.KFOUR) GO TO 7
    BT(I1+I4,K3+I)=DB3
    GO TO 7
 24 BT(I1,K1+I)=-IS*C9*DA2
    BT(I1,K4+I)=DB2
    BS(I1+I4,K1+I)=-IS*C9*(DA2+2./((1.-XNI)*EXXP))
    BS(I1+I4,K4+I)=DB2
    DO 9 J1=1,K1
    J=J1-1
    BS(I1+I4,J1)=BS(I1+I4,J1)-C10*(ALB(I,J,1,WZ,IN,NJ,NW,1,1)+
   1 ALB(I,J,4,WZ,IN,NJ,NW,1,1))
  9 CONTINUE
    IF(I.GT.KFOUR) GO TO 7
    BT(I1,K3+I)=DB3
    BS(I1+I4,K3+I)=DB3
    BS(I1+I4,I+1)=BS(I1+I4,I+1)-IS*C9*DB4+1./RR
    GO TO 7
 26 BS(I1,K1+I)=1.
    BS(I1+I4,K4+I)=DB2
    IF(I.GT.KFOUR) GO TO 7
    BS(I1+I4,K3+I)=DB3
    GO TO 7
 27 BT(I1,K1+I)=1.
    BT(I1+I4,K4+I)=DB2
    IF(I.GT.KFOUR) GO TO 7
    BT(I1+I4,K3+I)=DB3
    GO TO 7
 28 BT(I1,K1+I)=-IS*C9*DA2
    BT(I1,K4+I)=DB2
    BS(I1+I4,K1+I)=-IS*C9*(DA2+2./((1.-XNI)*EXXP))
    BS(I1+I4,K4+I)=DB2
    IF(I.GT.KFOUR) GO TO 7
    BT(I1,K3+I)=DB3
    BS(I1+I4,K3+I)=DB3
    GO TO 7
 30 BS(I1,K1+I)=1.
    BS(I1+I4,K4+I)=DB2
    DO 14 J1=1,KFOUR
    BT(I1+I4,J1+1)=BT(I1+I4,J1+1)-C10*(ALB(I,J1,2,WZP,IN,NJ,NW,1,1)+
   1ALB(I,J1,5,WZP,IN,NJ,NW,1,1))
 14 CONTINUE
```

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DYNAMIC STABILITY OF STRUCTURES: APPLICATION TO FRAMES, CYLINDR--ETC(U)
FEB 82 G J SIMITSES, I SHEINMAN F33615-79-C-3221
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3 of 3

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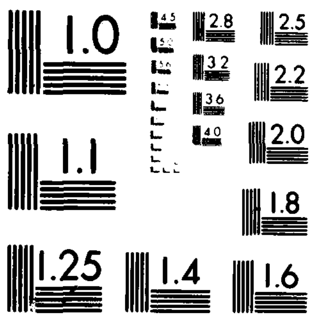

MICROCOPY RESOLUTION TEST CHART

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```
      BS(I1+I4,[illegible] BS(I1+I4,K3+I)+DB3
    7 CONTINUE
  666 RETURN
      END
      SUBROUTINE COEFF(EX,EY,XLAMD,YLAMD,RHOX,RHOY,ELAS)
      COMMON/GEOM/RR,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
      COMMON/FACT2/DL1,DL2,DL3,DL4,DA1,DA2,DA3,DA4,DB2,DB3,DB4,XNI,EXXP
      COMMON/CINTG/NEQPOT,MI(500)
      COMMON/FIDFR/DELTA,AL1,GA1,AL2,BT2,GA2
      COMMON/FOURIR/KFOUR,K6,K4,K3,K2,K1
      COMMON/FACT3/DL5,XL,XH
      K6=6*KFOUR+2
      K4=4*KFOUR+2
      K3=3*KFOUR+2
      K2=2*KFOUR
      K1=KFOUR+1
      XN2=XNI**2
      XH2=XH**2
      DALFA=(1.+XLAMD)*(1.+YLAMD)-XN2
      DD=ELAS*XH**3/(12.*(1.-XN2))
      EXXP=ELAS*XH/(1.-XN2)
      H11=1.+RHOX+12.*EX**2*XLAMD*(1.+YLAMD-XN2)/(XH2*DALFA)
      H22=1.+RHOY+12.*EY**2*YLAMD*(1.+XLAMD-XN2)/(XH2*DALFA)
      H12=1.+12.*XNI*EX*EY*XLAMD*YLAMD/(XH2*DALFA)
      Q11=XNI*EX*XLAMD/DALFA
      Q22=XNI*EY*YLAMD/DALFA
      Q12=-0.5*((1.+YLAMD)*EX*XLAMD+(1.+XLAMD)*EY*YLAMD)/DALFA
      D11=(1.+XLAMD)/(DALFA*EXXP)
      D22=(1.+YLAMD)/(DALFA*EXXP)
      D12=((1.+XLAMD)*(1.+YLAMD)-XNI)/(DALFA*EXXP*(1.-XNI))
      DL1=DD*H11
      DL2=DD*XNI*(1.+(EX*EY*XLAMD*YLAMD*12.)/(XH2*DALFA))
      DL3=EX*XLAMD*(1.+YLAMD)/DALFA
      DL4=-XNI*EX*XLAMD/DALFA
      DL5=DD*H22
      DA1=(1.+YLAMD)/(DALFA*EXXP)
      DA2=-XNI/(DALFA*EXXP)
      DA3=-(1.+YLAMD)*EX*XLAMD/DALFA
      DA4=XNI*EY*YLAMD/DALFA
      DB2=(1.+XLAMD)/(DALFA*EXXP)
      DB3=XNI*EX*XLAMD/DALFA
      DB4=-(1.+XLAMD)*EY*YLAMD/DALFA
      MI(1)=2*K6
       MI(NEQPOT)=2*K6
      NEQ1=NEQPOT-1
      DO 10 I1=2,NEQ1
      MI(I1)=K6
   10 CONTINUE
      DELTA=XL/(NEQPOT-1)
      AL1=-1./(2.*DELTA)
      GA1=1./(2.*DELTA)
      AL2=1./(DELTA**2)
      BT2=-2./DELTA**2
      GA2=1./DELTA**2
      RETURN
      END
      SUBROUTINE COEFNN(NNN)
      COMMON/GEOM/RR,DD,H11,H12,H22,Q11,Q12,Q22,D11,D12,D22
       COMMON/FACTOR/C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12
      C9=(NNN/RR)**2
      C1=DD*H11
      C2=2.*DD*H12*C9
      C3=2.*Q12*C9
      C4=DD*H22*C9**2
```

```
      C5=C11/[illegible]
      C6=1./(PR*D11)*C9
      C7=C9**2/(2.*D11)
      C8=G22*C9**2
      C10=Q9/2.
      C11=2.*Q9*D12
      C12=D22*C9**2
      RETURN
      END
      SUBROUTINE INVERT(NA,A,C,M,NM1,NM2,DET,IXP,IDET)
      DIMENSION A(NM1,NM1),C(NM2),M(NM2)
      DET=1.
      IXP=0
      NN=NA
      IF(NN.NE.1) GO TO 303
      DET=A(1,1)
      A(1,1)=1./A(1,1)
      GO TO 304
  303 DO 90 I=1,NN
   90 M(I)=-I
      DO 140 II=1,NN
      D=0.D0
      DO 112 K=1,NN
      IF(M(K)) 100,100,112
  100 DO 110 L=1,NN
      IF(M(L)) 103,103,110
  103 IF(ABS(D)-ABS(A(K,L))) 105,105,110
  105 LD=L
      KD=K
      D=A(K,L)
      BIGA=D
  110 CONTINUE
  112 CONTINUE
      IF(D.EQ.0.D0) GO TO 170
      GO TO 188
  170 WRITE(6,502)
      STOP
  502 FORMAT(/,5X,"DETERMINANT=0"/)
  188 NEMP=-M(LD)
      M(LD)=M(KD)
      M(KD)=NEMP
      DO 114 I=1,NN
      C(I)=A(I,LD)
      A(I,LD)=A(I,KD)
  114 A(I,KD)=0.D0
      A(KD,KD)=1.D0
      DO 115 J=1,NN
  115 A(KD,J)=A(KD,J)/D
      DO 135 I=1,NN
      IF(I.EQ.KD) GO TO 135
      DO 134 J=1,NN
      TEMP=C(I)*A(KD,J)
  134 A(I,J)=A(I,J)-TEMP
  135 CONTINUE
      IF(IDET.NE.1) GO TO 140
      DET=DET*BIGA
      IF(KD.NE.LD)DET=-DET
  629 IF(ABS(DET).LT.1.E+10) GO TO 630
      DET=DET/1.E+10
      IXP=IXP+10
      GO TO 629
  630 IF(ABS(DET).GT.1.E-10) GO TO 140
      DET=DET*1.E+10
      IXP=IXP-10
```

```
  140 CONTINUE
      DO 200 I=1,NN
      L=0
  150 L=L+1
      IF(M(L)-I) 150,160,150
  160 M(L)=M(I)
      M(I)=I
      DO 200 J=1,NN
      TEMP=A(L,J)
       A(L,J)=A(I,J)
  200 A(I,J)=TEMP
  304 RETURN
      END
      SUBROUTINE YMY(N1,A,B,C,N2,L1,L2,L3,T)
      DIMENSION A(L1,L2),B(L1,L1),C(L1,L2),T(L3)
      IF(N2.EQ.1) GO TO 100
      DO 11 I=1,N1
      DO 10 J=1,N2
      TEMP=0.
      DO 20 K=1,N1
   20 TEMP=TEMP+B(I,K)*C(K,J)
   10 T(J)=TEMP
      DO 30 J=1,N2
   30 A(I,J)=T(J)
   11 CONTINUE
      RETURN
  100 DO 111 I=1,N1
      TEMP=0.
      DO 120 K=1,N1
  120 TEMP=TEMP+B(I,K)*C(K,1)
  111 T(I)=TEMP
      DO 130 I=1,N1
  130 A(I,1)=T(I)
      RETURN
      END
      SUBROUTINE YSYMY(N2,N1,A,B,C,D,N3,L1,L2,L3,L4,T)
      DIMENSION A(L1,L3),B(L1,L3),C(L1,L2),D(L2,L3),T(L4)
       IF(N3.EQ.1) GO TO 100
      DO 11 I=1,N1
      DO 10 J=1,N3
      TEMP=0.
      DO 20 K=1,N2
   20 TEMP=TEMP+C(I,K)*D(K,J)
   10 T(J)=B(I,J)-TEMP
      DO 30 J=1,N3
   30 A(I,J)=T(J)
   11 CONTINUE
      RETURN
  100 DO 111 I=1,N1
      TEMP=0.
      DO 120 K=1,N2
  120 TEMP=TEMP+C(I,K)*D(K,1)
  111 T(I)=B(I,1)-TEMP
      DO 130 I=1,N1
  130 A(I,1)=T(I)
      RETURN
      END
       SUBROUTINE YMYM(V1,V2,A,N1,N2,M1,M2,M3,M4,T)
C     V1(1,N2)=V2(1,N1)*A(N1,N2)
       DIMENSION V1(1,M1),V2(1,M1),A(M2,M3),T(M4)
       DO 10 J=1,N2
       TEMP=0.
       DO 20 K=1,N1
   20  TEMP=TEMP+V2(1,K)*A(K,J)
```

```
   10  T(J)=TEMP
        DO 30 J=1,N2
   30  V1(1,J)=T(J)
       RETURN
       END
      SUBROUTINE POTERS(IDET,NRHS,MAXN,AP,BP,CP,GF,PR,XP,C,MT,T1,
     1 V1,MAX2,IXPM,DETM,XNXX,LN,NJ,NW,NF,LP,DP)
C    MAX2=MAXN*MAXN
C   CHANGED BY SHEINMAN IN OCT 1980 FOR MARIX WITH DP
C   LP IS THE COLUMN OF DP
C   LP=0 FOR THA REGULAR FORM WITHOUT DP
C   LP SUPPOSE TO BE GT.2 AND LT NEQPOT-1
C      JPS IS THE COLUMN NO. OF DP ON WHICH THE TERMS ARE NONZERO
        COMMON/NEWPT/JPS,INXXPX
      COMMON/CINTG/NEQPOT,MI(500)
      COMMON/CDISK/I21(501),I22(501),I23(501)
      DIMENSION AP(MAXN,MAXN),BP(MAXN,MAXN),CP(MAXN,MAXN)
      DIMENSION PR(MAXN,MAXN),GF(MAXN,NRHS),XP(MAXN,NRHS)
      DIMENSION T1(MAXN),C(MAXN),MT(MAXN),V1(MAX2)
         DIMENSION DP(MAXN,1),EP(52,1),VB1(1,52),VB2(1,52)
C     EQUIVALENCE (AP(1,1),V1(1))
          IF(LP.NE.1.AND.LP.NE.2.AND.LP.LT.NEQPOT-1)GO TO 70
        WRITE(6,71)LP
   71   FORMAT(//,2X,"LP=",I5,5X,"AND ITS SUPPOSE TO BE GT.2 AND LT
     1.NEQPOT-1")
          STOP
   70   CONTINUE
               R1=1.0
      IXPM=0
      DETM=1.
      DO 100 I=1,NEQPOT
      CALL ABCG(I,MAXN,CP,BP,AP,GF,NRHS,XNXX,LN,NJ,NW,NF,LP,DP)
      N=MI(I)
        IF(LP.EQ.0.OR.I.LT.LP+2) GO TO 80
          DO 4 K1=1,N
    4   GP(K1,1)=GP(K1,1)-DP(K1,1)*VG1
   80 IF(I.EQ.1) GO TO 888
      NMIN1=MI(I-1)
  888 IF(I.EQ.NEQPOT) GO TO 999
      NPLUS1=MI(I+1)
  999 CONTINUE
      IF(I.EQ.1) GO TO 12
         IF(LP.EQ.0.OR.I.LT.LP+2) GO TO 61
         DO 5 K1=1,N
         DO 5 K2=1,NMIN1
    5    CP(K1,K2)=CP(K1,K2)-DP(K1,1)*VB1(1,K2)*R1
         R1=-R1
   61 CALL YSYMY(NMIN1,N,BP,AP,CP,PR,N,MAXN,MAXN,MAXN,MAXN,T1)
   12 CALL INVERT(N,BP,C,MT,MAXN,MAXN,DET,IXP,IDET)
      IF(IDET.NE.1) GO TO 640
      DETM=DET*DETM
      IXPM=IXP+IXPM
      IF(ABS(DETM).LT.1.E+10) GO TO 630
      DETM=DETM/1.E+10
      IXPM=IXPM+10
      GO TO 640
  630 IF(ABS(DETM).GT.1.E-10) GO TO 640
      DETM=DETM*1.E+10
      IXPM=IXPM-10
  640 CONTINUE
      IF(I.EQ.NEQPOT) GO TO 102
         IF(I.NE.LP-1)GO TO 77
         DO 37 K1=1,N
         DO 37 K2=1,NMIN1
```

```
37     AP(K1,JPS)=AP(K1,JPS)-CP(K1,K2)*BP(K2,1)
77     CONTINUE
    CALL YMY(N,PR,BP,AP,NPLUS1,MAXN,MAXN,MAXN,T1)                      1213
    CALL XWRITE(21,PR,N,NPLUS1,MAXN,MAXN,I,MAX2,V1)                    1214
      IF(LP.EQ.0.OR.I.LT.LP)GO TO 102
      IF(I.GE.LP+2)GO TO 300
      IF(I.EQ.LP+1)GO TO 290
      DO 6 K1=1,NPLUS1
 6    VB1(1,K1)=PR(JPS,K1)
      GO TO 102
290   CALL YNYN(VB2,VB1,PR,N,NPLUS1,MAXN,MAXN,MAXN,MAXN,T1)
      KNS=NPLUS1
      GO TO 102
300   DO 7 K1=1,KNS
 7    VB1(1,K1)=VB2(1,K1)
      IF(I.EQ.NEQPOT-1)GO TO 102
      CALL YNYN(VB2,VB1,PR,N,NPLUS1,MAXN,MAXN,MAXN,MAXN,T1)
      KNS=NPLUS1
102 IF(I.EQ.1) GO TO 32                                                1215
    CALL YSYMY(NMIN1,N,XP,GP,CP,XP,NRHS,MAXN,MAXN,NRHS,MAXN,T1)        1216
    CALL YMY(N,XP,BP,XP,NRHS,MAXN,NRHS,MAXN,T1)                        1217
      IF(I.EQ.NEQPOT) GO TO 42
      IF(LP.EQ.0.OR.I.LT.LP) GO TO 90
      IF(I.GE.LP+2)GO TO 350
      IF(I.EQ.LP+1)GO TO 360
      VG1=XP(JPS,1)
      VG2=VG1
      GO TO 90
360   DO 19 K1=1,N
 19   VG2=VG2-VB1(1,K1)*XP(K1,1)
      GO TO 90
350   VG1=VG2
      IF(I.EQ.NEQPOT-1) GO TO 42
      DO 21 K1=1,N
 21   VG2=VG2+VB1(1,K1)*XP(K1,1)*R1
90     IF(LP.EQ.0.OR.I.GT.LP-2)GO TO 42
     CALL YSYMY(NMIN1,N,EP,DP,CP,EP,NRHS,MAXN,MAXN,NRHS,MAXN,T1)
     CALL YMY(N,EP,BP,EP,NRHS,MAXN,NRHS,MAXN,T1)
     CALL XWRITE(23,EP,N,NRHS,MAXN,NRHS,I,MAXN,T1)
    GO TO 42                                                           1218
 32 CALL YMY(N,XP,BP,GP,NRHS,MAXN,NRHS,MAXN,T1)                        1219
       IF(LP.EQ.0)GO TO 42
       CALL YMY(N,EP,BP,DP,NRHS,MAXN,NRHS,MAXN,T1)
       CALL XWRITE(23,EP,N,NRHS,MAXN,NRHS,I,MAXN,T1)
 42 CALL XWRITE(22,XP,N,NRHS,MAXN,NRHS,I,MAXN,T1)                      1220
100 CONTINUE                                                           1221
    MEQPOT=NEQPOT-1                                                    1222
    DO 200 K=1,MEQPOT                                                  1223
    NK=NEQPOT-K                                                        1224
    NMIN1=MI(NK)                                                       1225
    N=MI(NK+1)                                                         1226
    CALL XREAD(21,PR,NMIN1,N,MAXN,MAXN,NK,MAX2,V1)                     1227
    CALL XREAD(22,GP,NMIN1,NRHS,MAXN,NRHS,NK,MAXN,T1)                  1228
    CALL YSYMY(N,NMIN1,XP,GP,PR,XP,NRHS,MAXN,MAXN,NRHS,MAXN,T1)        1229
       IF(NK.NE.LP) GO TO 74
       VG1=XP(JPS,1)
 74   IF(LP.EQ.0.OR.NK.GT.LP-2)GO TO 92
       CALL XREAD(23,EP,NMIN1,NRHS,MAXN,NRHS,NK,MAXN,T1)
       DO 27 K1=1,NMIN1
 27   XP(K1,1)=XP(K1,1)-EP(K1,1)*VG1
 92 CALL XWRITE(22,XP,NMIN1,NRHS,MAXN,NRHS,NK,MAXN,T1)                 1230
200 CONTINUE                                                           1231
    RETURN                                                             1232
    END                                                                1233
```

```
      SUBROUTINE XREAD(ND,A,L1,L2,M1,M2,IND,M3,VV)
      COMMON/CDISK/I21(501),I22(501),I23(501)
      DIMENSION A(M1,M2),VV(M3)
C   RECORD  IND OF DIRECT ACCESS DATA SET ND IS READ AND ALLOCATED
C  BY ROWS INTO L1*L2 PORTION OF MATRIX A
       L3=L1*L2
      CALL READMS(ND,VV,L3,IND)
      KL=0
      DO 10 NROW=1,L1
      DO 10 NCOL=1,L2
      KL=KL+1
      A(NROW,NCOL)=VV(KL)
   10 CONTINUE
      RETURN
      END
      SUBROUTINE XWRITE(ND,A,L1,L2,M1,M2,IND,M3,VV)
      COMMON/CDISK/I21(501),I22(501),I23(501)
      DIMENSION A(M1,M2),VV(M3)
C  L1*L2 PORTION OF MATRIX A IS WRITTEN BY ROWS ON DIRECT ACCESS
C   DATA SET ND IN RECORD IND
      KL=0
      DO 10 NROW=1,L1
      DO 10 NCOL=1,L2
      KL=KL+1
      VV(KL)=A(NROW,NCOL)
   10 CONTINUE
      CALL WRITMS(ND,VV,KL,IND,-1)
      RETURN
      END
/EOR
 EXAMPLE 1 R/H=500 L/R=1. N=5
35,1,3,3,0,1,1
4.,4.,0.008,1350000.,0.3,1.0
0.,0.,0.,0.,0.,0.
2,2,2,18,1.
1.,5.,0.,15.,5.,90.,300.,60.
1,2,2,1
5,0,1
1
 EXAMPLE 1 R/H=500 L/R=1. N=6
35,1,3,3,0,1,1
4.,4.,0.008,1350000.,0.3,1.0
0.,0.,0.,0.,0.,0.
2,2,2,18,1.
1.,5.,0.,15.,5.,90.,300.,60.
1,2,2,1
6,0,1
1
 EXAMPLE 1 R/H=500 L/R=1. N=7
35,1,3,3,0,1,1
4.,4.,0.008,1350000.,0.3,1.0
0.,0.,0.,0.,0.,0.
2,2,2,18,1.
1.,5.,0.,15.,5.,90.,300.,60.
1,2,2,1
7,0,1
0
/EOR
```

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